

Some inequalities for partial derivatives on time scales

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Received: date / Accepted: date

Abstract We first prove some weighted inequalities for compositions of functions on time scales which are in turn applied to establish some new dynamic Opial-type inequalities in several variables. Some generalizations and applications to partial differential dynamic equations are also considered.

Keywords Opial's inequality · time scale · partial differential dynamic equation

Mathematics Subject Classification (2010) 26D15 · 26D10 · 26E70

1 Introduction

In 1960, Opial [16] established the following integral inequality

$$\int_0^b |f(x)f'(x)|dx \leq \frac{b}{4} \int_0^b |f'(x)|^2 dx, \quad (1.1)$$

where f is absolutely continuous on $[0, b]$ such that $f(0) = f(b) = 0$. In 1962, Beesack [4] showed the following result which implies (1.1) and is very useful in applications: If f is absolutely continuous on $[0, b]$ and $f(0) = 0$, then

$$\int_0^b |f(x)f'(x)|dx \leq \frac{b}{2} \int_0^b |f'(x)|^2 dx. \quad (1.2)$$

Since then, many generalizations of Opial's inequality (1.2) in various directions have been given, one of which was given in 1967 by Godunova and Levin [11]: Let F be a convex and increasing function on $[0, \infty)$ with $F(0) = 0$, and f be a real-valued absolutely continuous function defined on $[a, b]$ with $f(a) = 0$; then

$$\int_a^b F(|f(x)|)|f'(x)|dx \leq F\left(\int_a^b |f'(x)|dx\right). \quad (1.3)$$

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This work is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.32.

Later, some multidimensional generalizations of (1.3) were given, such as Pečarić [19], Pachpatte [18], and Andrić et al. [3].

In 2015, Duc, Nhan and Xuan [10] extended and generalized (1.2) to several independent variables and demonstrated the usefulness in the field of partial differential equations:

$$\begin{aligned} & \left[\int_{\Omega} \left| \partial^{\alpha} \left(\prod_{j=1}^m G_j(u_j(x)) \right) \right|^s K_{\alpha}(\mathbf{b}, \mathbf{x}) \sigma(\mathbf{x}) d\mathbf{x} \right]^{1/s} \\ & \leq C \prod_{j=1}^m \left[G_j \left(\int_{\Omega} |\partial^{\alpha} u_j(\mathbf{x})|^p K_{\alpha}(\mathbf{b}, \mathbf{x}) \rho_j(\mathbf{x}) d\mathbf{x} \right) \right]^{1/p}, \end{aligned} \quad (1.4)$$

where C is a constant.

In recent years, the theory of time scales which was introduced by Hilger [13] in order to unify the study of differential and difference equations, has received a lot of attention. The readers may find much of time scales calculus in books by Bohner and Peterson [5], [6]. One of main subjects of the qualitative analysis on time scales is to prove some new dynamic inequalities. Opial-type inequalities on time scales was first proved by Bohner and Kaymakçalan [7] in 2001 (see also [1]), in which they showed that if $f : [0, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $f(0) = 0$, then

$$\int_0^b |[f(x) + f^{\sigma}(x)]f^{\Delta}(x)| \Delta x \leq b \int_0^b |f^{\Delta}(x)|^2 \Delta x. \quad (1.5)$$

Afterwards, numerous authors have studied variants of (1.5) (see, for example, [14], [15], [20], [21], [22], [23], and [25]). The best reference here is the book by Agarwal, O'Regan, and Saker [2, Chapter 3], where the most popular articles on this subject are collected.

However, to the best of the author knowledge nothing is known regarding Opial-type inequalities involving functions of several variables and their partial derivatives on time scales. Thus, the aim of this paper is to study some weighted integral inequalities for delta derivatives acting on compositions of functions on time scales which are in turn applied to establish multidimensional dynamic Opial-type inequalities. As applications, we establish Lyapunov-type inequalities for half-linear dynamic equations and obtain upper bounds of solutions of certain integro-partial dynamic equations.

2 Preliminaries

In the most part we assume the readers were so familiar with basic time scales calculus. More information about time scales calculus can be found in [5] and [6]. In this section, we only present some basic definitions and notations about calculus in several variables on time scales.

Let \mathbb{T} be a time scale, and let σ, ρ , and Δ denote, respectively, the forward jump, backward jump, and delta operator on \mathbb{T} . Fix $n \in \mathbb{N}$ and let \mathbb{T}_j , where $j = 1, \dots, n$, be time scales, and

$$\Lambda^n = \mathbb{T}_1 \times \dots \times \mathbb{T}_n = \{\mathbf{x} = (x_1, \dots, x_n) : x_j \in \mathbb{T}_j \text{ for all } j \in [1, n]_{\mathbb{N}}\}$$

be the n -dimensional time scale. For $i \in [1, n]_{\mathbb{N}}$, let σ_j, ρ_j , and Δ_j denote, respectively, the forward jump, backward jump, and delta operator on \mathbb{T}_j . We define $\mathbb{T}_j^\kappa = \mathbb{T}_j$ if \mathbb{T}_j does not have a left-scattered maximum $x_{j_{\max}}$; otherwise $\mathbb{T}_j^\kappa = \mathbb{T}_j \setminus \{x_{j_{\max}}\}$. The graininess functions $\mu_j : \mathbb{T}_j \rightarrow [0, \infty)$ are defined by $\mu_j(x_j) = \sigma_j(x_j) - x_j$ for $j \in [1, n]_{\mathbb{N}}$. For $x_j \in \mathbb{T}_j, j \in [1, n]_{\mathbb{N}}$, we denote $\rho_j^2(x_j) = \rho_j(\rho_j(x_j))$ and $\rho_j^k(x_j) = \rho_j(\rho_j^{k-1}(x_j))$ for $k \in \mathbb{N}$. For convenience we put $\rho_j^0(x_j) = x_j, j \in [1, n]_{\mathbb{N}}$. For $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \Lambda^n$, we shall write $\mathbf{x} \leq \mathbf{y}$ instead of $x_j \leq y_j$ for all $j \in [1, n]_{\mathbb{N}}$. Analogously one has to understand $\mathbf{x} = \mathbf{y}, \mathbf{x} > \mathbf{y}$ and $\mathbf{x} < \mathbf{y}$, respectively. We put $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$. We denote by $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ the multi-index, i.e. $\lambda_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, j \in [1, n]_{\mathbb{N}}$. In particular, let $\mathbf{1} = (1, \dots, 1)$. Let $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n)$, and $\boldsymbol{\rho}^{\boldsymbol{\lambda}-\mathbf{1}}(\mathbf{b}) = (\rho_1^{\lambda_1-1}(b_1), \dots, \rho_n^{\lambda_n-1}(b_n))$ for $\boldsymbol{\lambda} \geq \mathbf{1}$, be in Λ^n such that $\mathbf{a} < \boldsymbol{\rho}^{\boldsymbol{\lambda}-\mathbf{1}}(\mathbf{b})$. Then we set

$$\begin{aligned}\Omega &= \{\mathbf{x} \in \Lambda^n : \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}, \\ \Omega^{\kappa^{\boldsymbol{\lambda}-\mathbf{1}}} &= \{\mathbf{x} \in \Lambda^n : \mathbf{a} \leq \mathbf{x} \leq \boldsymbol{\rho}^{\boldsymbol{\lambda}-\mathbf{1}}(\mathbf{b})\}, \\ \Omega_{\mathbf{x}} &= \{t \in \Lambda^n : \mathbf{a} \leq t \leq \mathbf{x}\}, \quad \mathbf{x} \in \Omega, \\ \bar{\Omega}_{\mathbf{x}} &= \{t \in \Lambda^n : \mathbf{x} \leq t \leq \boldsymbol{\rho}^{\boldsymbol{\lambda}-\mathbf{1}}(\mathbf{b})\}, \quad \mathbf{x} \in \Omega^{\kappa^{\boldsymbol{\lambda}-\mathbf{1}}}, \\ \Omega' &= [a_2, b_2]_{\mathbb{T}_2} \times \dots \times [a_n, b_n]_{\mathbb{T}_n}.\end{aligned}$$

For any real-valued rd-continuous function f defined on Ω we denote by $\int_{\Omega} f(\mathbf{x}) \Delta \mathbf{x}$, $\int_{\Omega^{\kappa^{\boldsymbol{\lambda}-\mathbf{1}}}} f(\mathbf{x}) \Delta \mathbf{x}$, $\int_{\Omega_{\mathbf{x}}} f(t) \Delta t$ for any $\mathbf{x} \in \Omega$, $\int_{\bar{\Omega}_{\mathbf{x}}} f(t) \Delta t$ for any $\mathbf{x} \in \Omega^{\kappa^{\boldsymbol{\lambda}-\mathbf{1}}}$, and $\int_{\Omega'} f(\mathbf{x}') \Delta \mathbf{x}'$ for $\mathbf{x}' \in \Omega'$, are n -fold integrals $\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} f(x_1, \dots, x_n) \Delta x_1 \dots \Delta x_n$, $\int_{a_1}^{\rho_1^{\lambda_1-1}(b_1)} \dots \int_{a_n}^{\rho_n^{\lambda_n-1}(b_n)} f(x_1, \dots, x_n) \Delta x_1 \dots \Delta x_n$, $\int_{a_1}^{x_1} \dots \int_{a_n}^{x_n} f(t_1, \dots, t_n) \Delta t_1 \dots \Delta t_n$, $\int_{x_1}^{\rho_1^{\lambda_1-1}(b_1)} \dots \int_{x_n}^{\rho_n^{\lambda_n-1}(b_n)} f(t_1, \dots, t_n) \Delta t_1 \dots \Delta t_n$, and $(n-1)$ -fold integral $\int_{a_2}^{b_2} \dots \int_{a_n}^{b_n} f(x_2, \dots, x_n) \Delta x_2 \dots \Delta x_n$, respectively.

Let $f : \Lambda^n \rightarrow \mathbb{R}$. The *partial delta derivative* of f with respect to $x_j \in \mathbb{T}_j^\kappa$ is defined as the limit

$$\lim_{\substack{t_j \rightarrow x_j \\ t_j \neq \sigma_j(x_j)}} \frac{f(x_1, \dots, \sigma_j(x_j), \dots, x_n) - f(x_1, \dots, t_j, \dots, x_n)}{\sigma_j(x_j) - t_j}$$

provided that this limit exists as a finite number, and is denoted by $\frac{\partial f(\mathbf{x})}{\Delta_j x_j}$. If f has partial derivatives $\frac{\partial f(\mathbf{x})}{\Delta_j x_j}, j \in [1, n]_{\mathbb{N}}$, then we can also consider their partial delta derivatives. These are called *second order* partial delta derivatives. We write

$$\frac{\partial^2 f(\mathbf{x})}{\Delta_j x_j^2} = \frac{\partial}{\Delta_j x_j} \left(\frac{\partial f(\mathbf{x})}{\Delta_j x_j} \right), \quad \frac{\partial^2 f(\mathbf{x})}{\Delta_j x_j \Delta_i x_i} = \frac{\partial}{\Delta_j x_j} \left(\frac{\partial f(\mathbf{x})}{\Delta_i x_i} \right).$$

Higher order partial delta derivatives are similarly defined.

Let $\boldsymbol{\lambda} \geq \mathbf{1}$ be a multi-index, we denote by $|\boldsymbol{\lambda}| = \lambda_1 + \dots + \lambda_n$; then we set

$$\frac{\partial^{\boldsymbol{\lambda}} f(\mathbf{x})}{\Delta \mathbf{x}^{\boldsymbol{\lambda}}} = \frac{\partial^{|\boldsymbol{\lambda}|} f(\mathbf{x})}{\Delta_1 x_1^{\lambda_1} \dots \Delta_n x_n^{\lambda_n}}.$$

By $C_{\text{rd}}^{n\lambda}(\Omega)$, we denote the set of all functions $f : \Omega \rightarrow \mathbb{R}$ which have rd-continuous derivatives $\frac{\partial^{k_1+\dots+k_j} f(\mathbf{x})}{\Delta_1 x_1^{k_1} \dots \Delta_j x_j^{k_j}}$ for $k_j \in [1, \lambda_j]_{\mathbb{N}}$, $j \in [1, n]_{\mathbb{N}}$. A function $\tau : \Omega \rightarrow \mathbb{R}$ is said to be a weight on Ω if τ is positive-valued and rd-continuous on Ω . Let us denote by $\mathcal{W}(\Omega)$ the set of all weights on Ω . Let $p \geq 1$ and $\tau \in \mathcal{W}(\Omega)$. We represent by $\mathcal{L}_a^p(\Omega, \tau, \lambda)$ the set of all functions $f : \Omega \rightarrow \mathbb{R}$ of class $C_{\text{rd}}^{n\lambda}(\Omega)$ for which $\frac{\partial^{k_j} f(\mathbf{x})}{\Delta_j x_j^{k_j}}|_{x_j=a_j} = 0$ for $k_j \in [0, \lambda_j - 1]_{\mathbb{N}}$, $j \in [1, n]_{\mathbb{N}}$, and that $\int_{\Omega} |\frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}}|^p \tau(\mathbf{x}) \Delta \mathbf{x} < \infty$.

We set

$$H_{\lambda}(\mathbf{x}, \mathbf{t}) = \prod_{j=1}^n h_{\lambda_j-1}^{(j)}(x_j, \sigma_j(t_j)), \quad \mathbf{x}, \mathbf{t} \in \Omega,$$

where $h_k^{(j)} : \mathbb{T}_j^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, is such that $h_0^{(j)}(t, s) \equiv 1$ for all $t, s \in \mathbb{T}_j$, and $h_{k+1}^{(j)}(t, s) = \int_s^t h_k^{(j)}(u, s) \Delta u$ for all $t, s \in \mathbb{T}_j$, $k \in \mathbb{N}_0$.

Let m be a positive integer and $0 < R \leq \infty$. We represent by \mathcal{H}_R^m the set of all functions $F : (-R, R)^m \rightarrow \mathbb{R}$ such that

1. $F \in C^1((-R, R)^m)$,
2. $F(0, \dots, 0) = 0$, and
3. $D_i F$ for $i \in [1, m]_{\mathbb{N}}$, are non-negative and increasing in each variable on $(0, R)$, where $D_i = \partial/\partial t_i$, $i \in [1, m]_{\mathbb{N}}$ for all $(t_1, \dots, t_m) \in (-R, R)^m$.

We give a preliminary lemma that we shall use in Section 3.

Lemma 2.1 *Let $F \in \mathcal{H}_R^m$ and $g_i : \mathbb{T} \rightarrow [0, R)$ for $i \in [1, m]_{\mathbb{N}}$, are delta differentiable on \mathbb{T}^{κ} such that g_i^{Δ} are non-negative on \mathbb{T}^{κ} ; then the composite function $F(g_1(x), \dots, g_m(x))$ is delta differentiable on \mathbb{T}^{κ} such that*

$$[F(g_1(x), \dots, g_m(x))]^{\Delta} \geq \sum_{i=1}^m D_i F(g_1(x), \dots, g_m(x)) g_i^{\Delta}(x), \quad x \in \mathbb{T}^{\kappa}. \quad (2.1)$$

Proof Fix $x \in \mathbb{T}^{\kappa}$ and put $t_i = g_i(x)$, $t_i^{\sigma} = g_i(\sigma(x))$, we see that $t_i \leq t_i^{\sigma}$ for $i \in [1, m]_{\mathbb{N}}$. We have two cases.

Case 1. Suppose that $x < \sigma(x)$. Then

$$[F(t_1, \dots, t_m)]^{\Delta} = \frac{F(t_1^{\sigma}, \dots, t_m^{\sigma}) - F(t_1, \dots, t_m)}{\sigma(x) - x}.$$

For all $i \in [1, m]_{\mathbb{N}}$, we set

$$A_i = \begin{cases} \frac{F(t_1, \dots, t_i^{\sigma}, \dots, t_m^{\sigma}) - F(t_1, \dots, t_i, t_{i+1}^{\sigma}, \dots, t_m^{\sigma})}{t_i^{\sigma} - t_i} & \text{if } t_i < t_i^{\sigma}, \\ 0 & \text{if } t_i = t_i^{\sigma}. \end{cases}$$

Then,

$$[F(t_1, \dots, t_m)]^{\Delta} = \sum_{i=1}^m A_i \frac{g_i(\sigma(x)) - g_i(x)}{\sigma(x) - x} = \sum_{i=1}^m A_i g_i^{\Delta}(x).$$

If $t_i = t_i^{\sigma}$, then $g_i^{\Delta}(x) = 0$; if $t_j < t_j^{\sigma}$ with $j \neq i$, then

$$A_j = \frac{F(t_1, \dots, t_j^{\sigma}, \dots, t_m^{\sigma}) - F(t_1, \dots, t_j, t_{j+1}^{\sigma}, \dots, t_m^{\sigma})}{t_j^{\sigma} - t_j} = D_j F(t_1, \dots, c_j, t_{j+1}^{\sigma}, \dots, t_m^{\sigma}),$$

by the mean value theorem, where $c_j \in (t_j, t_j^\sigma)$. Since $D_j F$, $i \in [1, m]_{\mathbb{N}}$, are increasing in each variable on $(0, R)$, we have

$$D_j F(t_1, \dots, c_j, t_{j+1}^\sigma, \dots, t_m^\sigma) \geq D_j F(t_1, \dots, t_m) = D_j F(g_1(x), \dots, g_m(x)),$$

which yields (2.1).

Case 2. Suppose that $x = \sigma(x)$. We set

$$F_i(g(s)) := F(g_1(x), \dots, g_{i-1}(x), g_i(s), g_{i+1}(s), \dots, g_m(s)),$$

$$F_i(g(x)) := F(g_1(x), \dots, g_{i-1}(x), g_i(x), g_{i+1}(s), \dots, g_m(s)), \quad i \in [1, m]_{\mathbb{N}}.$$

We have

$$[F(g_1(x), \dots, g_m(x))]^\Delta = \lim_{s \rightarrow x} \frac{F(g_1(s), \dots, g_m(s)) - F(g_1(x), \dots, g_m(x))}{s - x},$$

where

$$\frac{F(g_1(s), \dots, g_m(s)) - F(g_1(x), \dots, g_m(x))}{s - x} = \sum_{i=1}^m \frac{g_i(s) - g_i(x)}{s - x} \frac{F_i(g(s)) - F_i(g(x))}{g_i(s) - g_i(x)}.$$

By the mean value theorem, there exist $\xi_i(s), i \in [1, m]_{\mathbb{N}}$, which are between $g_i(x)$ and $g_i(s)$, such that

$$\frac{F_i(g(s)) - F_i(g(x))}{g_i(s) - g_i(x)} = D_i F(g_1(x), \dots, g_{i-1}(x), \xi_i(s), g_{i+1}(s), \dots, g_m(s)).$$

Since g_i for $i \in [1, m]_{\mathbb{N}}$, are delta differentiable on \mathbb{T}^κ , then g_i for $i \in [1, m]_{\mathbb{N}}$, are continuous at x . Therefore, $\lim_{s \rightarrow x} \xi_i(s) = g_i(x)$ and $\lim_{s \rightarrow x} g_i(s) = g_i(x)$ for $i \in [1, m]_{\mathbb{N}}$, which gives us the desired result. \square

3 Integral inequalities on time scales

Theorem 3.1 Let $F \in \mathcal{H}_R^m$ for $0 < R \leq \infty$, and $f_i : \Omega \rightarrow (-R, R)$ which satisfies $\int_\Omega \left| \frac{\partial^1 f_i(x)}{\Delta x^1} \right| \Delta x < R$ for $i \in [1, m]_{\mathbb{N}}$. If $f_i \in \mathcal{L}_a^1(\Omega, 1, \mathbf{1})$ for all $i \in [1, m]_{\mathbb{N}}$, then

$$\begin{aligned} & \int_\Omega \left(\sum_{i=1}^m D_i F(|f_1(x)|, \dots, |f_m(x)|) \left| \frac{\partial^1 f_i(x)}{\Delta x^1} \right| \right) \Delta x \\ & \leq F \left(\int_\Omega \left| \frac{\partial^1 f_1(x)}{\Delta x^1} \right| \Delta x, \dots, \int_\Omega \left| \frac{\partial^1 f_m(x)}{\Delta x^1} \right| \Delta x \right). \end{aligned} \quad (3.1)$$

Proof Since $f_i \in \mathcal{L}_a^1(\Omega, 1, \mathbf{1})$, it follows that

$$f_i(x) = \int_{\Omega_x} \frac{\partial^1 f_i(t)}{\Delta t^1} \Delta t \quad (3.2)$$

for all $x \in \Omega$ and all $i \in [1, m]_{\mathbb{N}}$. Let

$$g_i(x_1) := \int_{a_1}^{x_1} \int_{\Omega'} \left| \frac{\partial^1 f_i(t)}{\Delta t^1} \right| \Delta t \quad \text{for } x_1 \in [a_1, b_1]_{\mathbb{T}}, \quad i \in [1, m]_{\mathbb{N}}.$$

We see that $|f_i(\mathbf{x})| \leq g_i(x_1)$ for $\mathbf{x} \in \Omega$ and functions $g_i, i \in [1, m]_{\mathbb{N}}$, are increasing on $[a_1, b_1]_{\mathbb{T}}$. By $F \in \mathcal{H}_R^m$, we obtain

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^m D_i F(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|) \left| \frac{\partial^1 f_i(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \right) \Delta \mathbf{x} \\ & \leq \int_{\Omega} \left(\sum_{i=1}^m D_i F(g_1(x_1), \dots, g_m(x_1)) \left| \frac{\partial^1 f_i(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \right) \Delta \mathbf{x} \\ & \leq \int_{a_1}^{b_1} \left(\sum_{i=1}^m D_i F(g_1(x_1), \dots, g_m(x_1)) \int_{\Omega'} \left| \frac{\partial^1 f_i(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x}' \right) \Delta x_1 \\ & \leq \int_{a_1}^{b_1} \left(\sum_{i=1}^m D_i F(g_1(x_1), \dots, g_m(x_1)) \frac{\partial g_i(x_1)}{\Delta_1 x_1} \right) \Delta x_1, \end{aligned}$$

which, in view of Lemma 2.1, yields

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^m D_i F(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|) \left| \frac{\partial^1 f_i(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \right) \Delta \mathbf{x} \\ & \leq \int_{a_1}^{b_1} F^{\Delta_1}(g_1(x_1), \dots, g_m(x_1)) \Delta x_1 \\ & = F \left(\int_{\Omega} \left| \frac{\partial^1 f_1(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x}, \dots, \int_{\Omega} \left| \frac{\partial^1 f_m(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \right), \end{aligned}$$

which completes the proof. \square

Remark 3.1 From Theorem 3.1 we can obtain many known results.

1. If $\mathbb{T} = \mathbb{R}$, then Theorem 3.1 becomes [8, Theorem 1] which was established by Brnetić and Pečarić.
2. If $\mathbb{T} = \mathbb{R}$, $n = m = 1$, and F is convex on $[0, \infty)$, then inequality (3.1) reduces to (1.3).
3. Let $\mathbb{T} = \mathbb{R}$, $n = 1$, and $F(x_1, \dots, x_m) = |x_1 \cdots x_m|$; then inequality (3.1) becomes [18, Theorem 1].

The following theorem is a generalization of Theorem 3.1.

Theorem 3.2 Let $F \in \mathcal{H}_R^m$ for $0 < R \leq \infty$, and $f_i : \Omega \rightarrow (-R, R)$ which satisfies $H_{\lambda}(\mathbf{b}, \mathbf{a}) \int_{\Omega} \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \Delta \mathbf{x} < R$ for $i \in [1, m]_{\mathbb{N}}$. If $f_i \in \mathcal{L}_a^1(\Omega, 1, \lambda)$ for all $i \in [1, m]_{\mathbb{N}}$, then

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^m D_i F(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|) \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \right) \Delta \mathbf{x} \\ & \leq \frac{1}{H_{\lambda}(\mathbf{b}, \mathbf{a})} F \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \int_{\Omega} \left| \frac{\partial^{\lambda} f_1(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \Delta \mathbf{x}, \dots, H_{\lambda}(\mathbf{b}, \mathbf{a}) \int_{\Omega} \left| \frac{\partial^{\lambda} f_m(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \Delta \mathbf{x} \right). \end{aligned} \quad (3.3)$$

Proof By $f_i \in \mathcal{L}_a^1(\Omega, 1, \lambda)$, $i \in [1, m]_{\mathbb{N}}$, and Taylor's formula [12], we have

$$f_i(\mathbf{x}) = \int_{\Omega_{\mathbf{x}}} H_{\lambda}(\mathbf{x}, \mathbf{t}) \frac{\partial^{\lambda} f_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}} \Delta \mathbf{t}, \quad \mathbf{x} \in \Omega.$$

Therefore,

$$|f_i(x)| \leq \int_{\Omega_x} H_\lambda(x, t) \left| \frac{\partial^\lambda f_i(t)}{\Delta t^\lambda} \right| \Delta t \leq H_\lambda(b, a) \int_{\Omega_x} \left| \frac{\partial^\lambda f_i(t)}{\Delta t^\lambda} \right| \Delta t, \quad x \in \Omega.$$

Now, we define

$$u_i(x) := H_\lambda(b, a) \int_{\Omega_x} \left| \frac{\partial^\lambda f_i(t)}{\Delta t^\lambda} \right| \Delta t, \quad x \in \Omega, \quad i \in [1, m]_{\mathbb{N}},$$

we see that $|f_i(x)| \leq u_i(x)$ for $x \in \Omega$ and $u_i(x)$ are increasing in each variable and

$$\left| \frac{\partial^\lambda f_i(x)}{\Delta x^\lambda} \right| = \frac{1}{H_\lambda(b, a)} \frac{\partial^1 u_i(x)}{\Delta x^1}, \quad x \in \Omega, \quad i \in [1, m]_{\mathbb{N}}.$$

Since $D_i F$ for $i \in [1, m]_{\mathbb{N}}$, are non-negative, continuous and increasing in each variable on $(0, R)$, we have

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^m D_i F(|f_1(x)|, \dots, |f_m(x)|) \left| \frac{\partial^\lambda f_i(x)}{\Delta x^\lambda} \right| \right) \Delta x \\ & \leq \frac{1}{H_\lambda(b, a)} \int_{\Omega} \left(\sum_{i=1}^m D_i F(u_1(x), \dots, u_m(x)) \frac{\partial^1 u_i(x)}{\Delta x^1} \right) \Delta x, \end{aligned}$$

which, in view of (3.1), gives (3.3). \square

Remark 3.2 Let $\mathbb{T} = \mathbb{R}$; then Theorem 3.2 becomes [3, Theorem 2.1] which was established by Andrić.

Next, for $0 < R \leq \infty$, we denote by $\mathcal{G}_R^{1,m}$ the class of all functions $G : (-R, R)^m \rightarrow \mathbb{R}$ satisfying the following conditions:

- (i) $G \in C^1((-R, R)^m)$,
- (ii) $G(0, \dots, 0) = 0$, and
- (iii) if $x_i \leq y_i^{1/p} z_i^{1/q}$, $0 < x_i, y_i, z_i < R$ for $i \in [1, m]_{\mathbb{N}}$, then $0 \leq D_i G(x_1, \dots, x_m) \leq [D_i G(y_1, \dots, y_m)]^{1/p} [D_i G(z_1, \dots, z_m)]^{1/q}$, where p, q are conjugate exponents $1/p + 1/q = 1$.

Remark 3.3 If $G \in \mathcal{G}_R^{1,m}$, then $G \in \mathcal{H}_R^m$.

Proof Let $G \in \mathcal{G}_R^{1,m}$. For each $i \in [1, m]_{\mathbb{N}}$ and $0 < x_i \leq y_i < R$, from (iii) we have

$$\begin{aligned} 0 \leq D_i G(x_1, \dots, x_i, \dots, x_m) & \leq [D_i G(y_1, \dots, y_i, \dots, y_m)]^{1/p} [D_i G(y_1, \dots, y_i, \dots, y_m)]^{1/q} \\ & = D_i G(y_1, \dots, y_i, \dots, y_m). \end{aligned}$$

Therefore, $G \in \mathcal{H}_R^m$.

Example 3.1 The functions $G(x_1, \dots, x_m) = |x_1|^{\gamma_1} \text{sign}(x_1) \cdots |x_m|^{\gamma_m} \text{sign}(x_m)$, $H(x_1, \dots, x_m) = |x_1|^{\gamma_1} + \cdots + |x_m|^{\gamma_m}$ for $\gamma_i \geq 1$ for all $i \in [1, m]_{\mathbb{N}}$ are in $\mathcal{G}_\infty^{1,m}$.

From on now, we always assume that $\alpha, \beta > 0$ and $\alpha + \beta > 1$ and $G \in \mathcal{G}_R^{1,m}$. We have the following result.

Theorem 3.3 Let $\omega_i, \tau_i, \in \mathcal{W}(\Omega)$, and $f_i : \Omega \rightarrow (-R, R)$ be such that

$$\int_{\Omega} \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau_i(\mathbf{x}) \Delta \mathbf{x} < R \quad \text{for } i \in [1, m]_{\mathbb{N}}.$$

If $f_i \in \mathcal{L}_a^{\alpha+\beta}(\Omega, \tau_i, \lambda)$ for all $i \in [1, m]_{\mathbb{N}}$ and

$$K_{\Omega} := \left[\int_{\Omega} \left(\sum_{i=1}^m [D_i G(V_1(\mathbf{x}), \dots, V_m(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\beta}} \omega_i^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau_i^{-\frac{\alpha}{\beta}}(\mathbf{x}) \right) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} < \infty, \quad (3.4)$$

where

$$V_i(\mathbf{x}) := \int_{\Omega_x} \left(H_{\lambda}(\mathbf{x}, t) \right)^{\frac{\alpha+\beta}{\alpha+\beta-1}} (\tau_i(t))^{\frac{1}{1-\alpha-\beta}} \Delta t$$

for $\mathbf{x} \in \Omega, i \in [1, m]_{\mathbb{N}}$, then

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^m [D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|)]^{\alpha} \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha} \omega_i(\mathbf{x}) \right) \Delta \mathbf{x} \\ & \leq K_{\Omega} \left[G \left(\int_{\Omega} \left| \frac{\partial^{\lambda} f_1(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau_1(\mathbf{x}) \Delta \mathbf{x}, \dots, \int_{\Omega} \left| \frac{\partial^{\lambda} f_m(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau_m(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}}. \end{aligned} \quad (3.5)$$

Proof As in the proof of Theorem 3.2, for any $\mathbf{x} \in \Omega$ and all $i \in [1, m]_{\mathbb{N}}$, we have

$$|f_i(\mathbf{x})| \leq \int_{\Omega_x} H_{\lambda}(\mathbf{x}, t) \left| \frac{\partial^{\lambda} f_i(t)}{\Delta t^{\lambda}} \right| \Delta t. \quad (3.6)$$

Applying Hölder's inequality with indices $(\alpha+\beta)/(\alpha+\beta-1)$ and $(\alpha+\beta)$ to (3.6), we get

$$\begin{aligned} |f_i(\mathbf{x})| & \leq \left(\int_{\Omega_x} \left(H_{\lambda}(\mathbf{x}, t) \right)^{\frac{\alpha+\beta}{\alpha+\beta-1}} (\tau_i(t))^{\frac{1}{1-\alpha-\beta}} \Delta t \right)^{\frac{\alpha+\beta-1}{\alpha+\beta}} \\ & \quad \times \left(\int_{\Omega_x} \left| \frac{\partial^{\lambda} f_i(t)}{\Delta t^{\lambda}} \right|^{\alpha+\beta} \tau_i(t) \Delta t \right)^{\frac{1}{\alpha+\beta}} \\ & =: \left(V_i(\mathbf{x}) \right)^{\frac{\alpha+\beta-1}{\alpha+\beta}} \left(U_i(\mathbf{x}) \right)^{\frac{1}{\alpha+\beta}}, \end{aligned} \quad (3.7)$$

where

$$U_i(\mathbf{x}) = \int_{\Omega_x} \left| \frac{\partial^{\lambda} f_i(t)}{\Delta t^{\lambda}} \right|^{\alpha+\beta} \tau_i(t) \Delta t, \quad \mathbf{x} \in \Omega, \quad i \in [1, m]_{\mathbb{N}}.$$

Thus, since $G \in \mathcal{G}_R^{1,m}$ from (3.7) we obtain

$$\begin{aligned} D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|) & \leq [D_i G(V_1(\mathbf{x}), \dots, V_m(\mathbf{x}))]^{\frac{\alpha+\beta-1}{\alpha+\beta}} \\ & \quad \times [D_i G(U_1(\mathbf{x}), \dots, U_m(\mathbf{x}))]^{\frac{1}{\alpha+\beta}}; \end{aligned}$$

hence

$$\begin{aligned}
& \sum_{i=1}^m [D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|)]^\alpha \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^\alpha \omega_i(\mathbf{x}) \\
& \leq \sum_{i=1}^m [D_i G(V_1(\mathbf{x}), \dots, V_m(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\alpha+\beta}} \\
& \quad \times [D_i G(U_1(\mathbf{x}), \dots, U_m(\mathbf{x}))]^{\frac{\alpha}{\alpha+\beta}} \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^\alpha \omega_i(\mathbf{x}).
\end{aligned} \tag{3.8}$$

By applying Hölder's inequality for sum with indices $(\alpha + \beta)/\beta$ and $(\alpha + \beta)/\alpha$ to (3.8), we get

$$\begin{aligned}
& \sum_{i=1}^m [D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|)]^\alpha \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^\alpha \omega_i(\mathbf{x}) \\
& \leq \left(\sum_{i=1}^m [D_i G(V_1(\mathbf{x}), \dots, V_m(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\beta}} \omega_i^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau_i^{-\frac{\alpha}{\beta}}(\mathbf{x}) \right)^{\frac{\beta}{\alpha+\beta}} \\
& \quad \times \left(\sum_{i=1}^m D_i G(U_1(\mathbf{x}), \dots, U_m(\mathbf{x})) \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_i(\mathbf{x}) \right)^{\frac{\alpha}{\alpha+\beta}}.
\end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) with respect to \mathbf{x} over Ω and using Hölder's inequality with indices $(\alpha + \beta)/\beta$ and $(\alpha + \beta)/\alpha$ give

$$\begin{aligned}
& \int_{\Omega} \left(\sum_{i=1}^m [D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|)]^\alpha \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^\alpha \omega_i(\mathbf{x}) \right) \Delta \mathbf{x} \\
& \leq \left[\int_{\Omega} \left(\sum_{i=1}^m [D_i G(V_1(\mathbf{x}), \dots, V_m(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\beta}} \omega_i^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau_i^{-\frac{\alpha}{\beta}}(\mathbf{x}) \right) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} \\
& \quad \times \left[\int_{\Omega} \left(\sum_{i=1}^m D_i G(U_1(\mathbf{x}), \dots, U_m(\mathbf{x})) \frac{\partial^1 U_i(\mathbf{x})}{\Delta \mathbf{x}^1} \right) \Delta \mathbf{x} \right]^{\frac{\alpha}{\alpha+\beta}},
\end{aligned} \tag{3.10}$$

which, in view of (3.1), yields (3.5). This concludes the proof. \square

Remark 3.4 1. In the special case when $n = m = 1$, $G(x) = |x|^{(\alpha+\beta)/\alpha}$, $\omega_1 = \tau_1 \equiv 1$, $\lambda = \lambda_1$, then inequality (3.5) becomes [15, Theorem 3.2].

2. If $n = \lambda = \alpha = 1$, $m = 2$, $G(x_1, x_2) = |x_1 x_2|$, $\beta = p - 1$, for $1 < p \leq 2$, $\omega_1 = \psi^\sigma$, $\tau_1 = \phi[\psi^\sigma]^{p/2}$, where $\psi^\sigma = \psi \circ \sigma$, $\phi, \psi \in \mathcal{W}([a, b]_{\mathbb{T}})$ and ψ is decreasing on $[a, b]_{\mathbb{T}}$, then inequality (3.5) reduces to

$$\begin{aligned}
& \int_a^b \psi^\sigma(x) (|f_1^\Delta(x) f_2(x)| + |f_1(x) f_2^\Delta(x)|) \Delta x \\
& \leq \frac{K(a, b)}{2^{2/p}} \left[\int_a^b (|f_1^\Delta(x)|^p + |f_2^\Delta(x)|^p) \phi(x) [\psi^\sigma(x)]^{p/2} \Delta x \right]^{2/p},
\end{aligned}$$

where

$$\begin{aligned}
K(a, b) &= \left[2 \int_a^b \frac{[\psi^\sigma(x)]^{q/2}}{\phi^{q/p}(x)} \left(\int_a^x \frac{\Delta t}{\phi^{q/p}(t) [\psi^\sigma(t)]^{q/2}} \right) \Delta x \right]^{1/q} \\
&\leq \left[2 \int_a^b \frac{1}{\phi^{q/p}(x)} \left(\int_a^x \frac{\Delta t}{\phi^{q/p}(t)} \right) \Delta x \right]^{1/q} \\
&\leq \left[\int_a^b \frac{\Delta x}{\phi^{q/p}(x)} \right]^{2/q}, \tag{3.11}
\end{aligned}$$

where p, q are conjugate exponents. Therefore, Theorem 3.3 improves and generalizes [25, Theorem 3.1].

3. Note that when $\mathbb{T} = \mathbb{R}$, $n = m = 2, \alpha = s \geq 1, \beta = 2r + s$ for $r \geq 0$, $G(x_1, x_2) = |x_1 x_2|^{(r+s)/s}, \omega_1 = \omega_2 = \tau_1 = \tau_2 = \omega$, where ω is decreasing in each variables, we see that inequality (3.5) improves [9, Theorem 1].

From Theorem 3.3 we have the following result.

Corollary 3.1 *Assume that conditions in Theorem 3.3 hold, then*

$$\begin{aligned}
&\int_{\Omega} \left(\sum_{i=1}^m \left[D_i G \left(\left| \frac{\partial^{\xi_1} f_1(\mathbf{x})}{\Delta \mathbf{x}^{\xi_1}} \right|, \dots, \left| \frac{\partial^{\xi_m} f_m(\mathbf{x})}{\Delta \mathbf{x}^{\xi_m}} \right| \right) \right]^{\alpha} \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha} \omega_i(\mathbf{x}) \right) \Delta \mathbf{x} \\
&\leq \hat{K}_{\Omega} \left[G \left(\int_{\Omega} \left| \frac{\partial^{\lambda} f_1(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau_1(\mathbf{x}) \Delta \mathbf{x}, \dots, \int_{\Omega} \left| \frac{\partial^{\lambda} f_m(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau_m(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}} \tag{3.12}
\end{aligned}$$

for $\xi_i \leq \lambda, i \in [1, n]_{\mathbb{N}}$, where \hat{K}_{Ω} is obtained by substituting

$$V_i(\mathbf{x}) := \int_{\Omega_{\mathbf{x}}} (H_{\lambda-\xi_i}(\mathbf{x}, \mathbf{t}))^{\frac{\alpha+\beta}{\alpha+\beta-1}} (\tau_i(\mathbf{t}))^{\frac{1}{1-\alpha-\beta}} \Delta \mathbf{t}$$

into (3.4).

Remark 3.5 Inequality (3.12) is the same as [21, Theorem 2.10], if we take $n = m = 1, G(x) = |x|^{\gamma}, \gamma \geq 1$.

The following theorem is similar to Theorem 3.3.

Theorem 3.4 *Let $\omega_i, \tau_i \in \mathcal{W}(\Omega^{\kappa^{\lambda-1}})$, and $f_i : \Omega \rightarrow (-R, R)$ which satisfies*

$$\int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau_i(\mathbf{x}) \Delta \mathbf{x} < R \quad \text{for } i \in [1, m]_{\mathbb{N}}.$$

If $f_i \in \mathcal{L}_{\rho^{\lambda-1}(b)}^{\alpha+\beta}(\Omega^{\kappa^{\lambda-1}}, \tau_i, \lambda)$ for all $i \in [1, m]_{\mathbb{N}}$ and

$$K_{\Omega^{\kappa^{\lambda-1}}}^* := \left[\int_{\Omega^{\kappa^{\lambda-1}}} \left(\sum_{i=1}^m [D_i G(V_1^*(\mathbf{x}), \dots, V_m^*(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\beta}} \omega_i^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau_i^{-\frac{\alpha}{\beta}}(\mathbf{x}) \right) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} \tag{3.13}$$

is finite, where

$$V_i^*(\mathbf{x}) := \int_{\Omega_{\mathbf{x}}} |H_{\lambda}(\mathbf{x}, \mathbf{t})|^{\frac{\alpha+\beta}{\alpha+\beta-1}} (\tau_i(\mathbf{t}))^{\frac{1}{1-\alpha-\beta}} \Delta \mathbf{t}$$

for $\mathbf{x} \in \Omega^{\kappa^{\lambda-1}}$, $i \in [1, m]_{\mathbb{N}}$, then

$$\begin{aligned} \int_{\Omega^{\kappa^{\lambda-1}}} \left(\sum_{i=1}^m [D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|)]^\alpha \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^\alpha \omega_i(\mathbf{x}) \right) \Delta \mathbf{x} &\leq K_{\Omega^{\kappa^{\lambda-1}}}^* \\ &\times \left[G \left(\int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^\lambda f_1(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_1(\mathbf{x}) \Delta \mathbf{x}, \dots, \int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^\lambda f_m(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_m(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}}. \end{aligned} \quad (3.14)$$

Proof For all $i \in [1, m]_{\mathbb{N}}$, and $\mathbf{x} \in \Omega^{\kappa^{\lambda-1}}$, we have

$$f_i(\mathbf{x}) = (-1)^n \int_{\bar{\Omega}_{\mathbf{x}}} H_\lambda(\mathbf{x}, t) \frac{\partial^\lambda f_i(t)}{\Delta t^\lambda} \Delta t.$$

In the proof of Theorem 3.3, replacing $\Omega_{\mathbf{x}}$ by $\bar{\Omega}_{\mathbf{x}}$ we get (3.14). \square

Remark 3.6 Since (3.11), note that when $n = \lambda = \alpha = 1$, $\beta = p - 1$, for $1 < p \leq 2$, $m = 2$, $G(x_1, x_2) = |x_1 x_2|$, $\omega_1 = \psi^\sigma$, $\tau_1 = \phi[\psi^\sigma]^{p/2}$, where $\phi, \psi \in \mathcal{W}([a, b]_{\mathbb{T}})$ and ψ is decreasing on $[a, b]_{\mathbb{T}}$, we see that inequality (3.14) improves [25, Theorem 3.2].

By applying Theorems 3.3 and 3.4 on $\Omega_{\mathbf{c}}$ and $\bar{\Omega}_{\mathbf{c}}$, respectively, with $\mathbf{c} \in \Omega^{\kappa^{\lambda-1}}$ is such that $\Omega^{\kappa^{\lambda-1}} = \Omega_{\mathbf{c}} \cup \bar{\Omega}_{\mathbf{c}}$, and summing the resulting inequalities, we have the following result.

Theorem 3.5 Let $\omega_i, \tau_i \in \mathcal{W}(\Omega^{\kappa^{\lambda-1}})$, and $f_i : \Omega \rightarrow (-R, R)$ be such that

$$\int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_i(\mathbf{x}) \Delta \mathbf{x} < R \quad \text{for } i \in [1, m]_{\mathbb{N}}.$$

If $f_i \in \mathcal{L}_a^{\alpha+\beta}(\Omega_{\mathbf{c}}, \tau_i, \lambda) \cap \mathcal{L}_{\rho^{\lambda-1}(b)}^{\alpha+\beta}(\bar{\Omega}_{\mathbf{c}}, \tau_i, \lambda)$ for all $i \in [1, m]_{\mathbb{N}}$, then we have

$$\begin{aligned} \int_{\Omega^{\kappa^{\lambda-1}}} \left(\sum_{i=1}^m [D_i G(|f_1(\mathbf{x})|, \dots, |f_m(\mathbf{x})|)]^\alpha \left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^\alpha \omega_i(\mathbf{x}) \right) \Delta \mathbf{x} \\ \leq K_{\Omega_{\mathbf{c}}} \left[G \left(\int_{\Omega_{\mathbf{c}}} \left| \frac{\partial^\lambda f_1(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_1(\mathbf{x}) \Delta \mathbf{x}, \dots, \int_{\Omega_{\mathbf{c}}} \left| \frac{\partial^\lambda f_m(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_m(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}} \\ + K_{\bar{\Omega}_{\mathbf{c}}}^* \left[G \left(\int_{\bar{\Omega}_{\mathbf{c}}} \left| \frac{\partial^\lambda f_1(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_1(\mathbf{x}) \Delta \mathbf{x}, \dots, \int_{\bar{\Omega}_{\mathbf{c}}} \left| \frac{\partial^\lambda f_m(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|^{\alpha+\beta} \tau_m(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}} \end{aligned} \quad (3.15)$$

for all $\mathbf{c} \in \Omega^{\kappa^{\lambda-1}}$ is such that $\Omega^{\kappa^{\lambda-1}} = \Omega_{\mathbf{c}} \cup \bar{\Omega}_{\mathbf{c}}$, where $K_{\Omega_{\mathbf{c}}}, K_{\bar{\Omega}_{\mathbf{c}}}^*$ are defined as in (3.4) and (3.13), respectively.

Remark 3.7 If $n = \lambda = \alpha = 1$, $\beta = p - 1$, for $1 < p \leq 2$, $m = 2$, $G(x_1, x_2) = |x_1 x_2|$, $\omega_1 = \psi^\sigma$, $\tau_1 = \phi[\psi^\sigma]^{p/2}$, where $\phi, \psi \in \mathcal{W}([a, b]_{\mathbb{T}})$ and ψ is decreasing on $[a, b]_{\mathbb{T}}$, we see that inequality (3.15) improves [25, Theorem 3.2].

Let us mention some important consequences of Theorems 3.3, 3.4, and 3.5. First, let $m = 1$; then we obtain the following corollary.

Corollary 3.2 Let $G \in \mathcal{G}_R^{1,1}$, $\omega, \tau \in \mathcal{W}(\Omega)$, and $f : \Omega \rightarrow (-R, R)$ be such that

$$\int_{\Omega} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau(\mathbf{x}) \Delta \mathbf{x} < R.$$

If $f \in \mathcal{L}_a^{\alpha+\beta}(\Omega, \tau, \lambda)$ and

$$N_{\Omega} := \left[\int_{\Omega} [G'(\vartheta(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\beta}} \omega^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau^{-\frac{\alpha}{\beta}}(\mathbf{x}) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} < \infty,$$

where

$$\vartheta(\mathbf{x}) := \int_{\Omega_{\mathbf{x}}} \left(H_{\lambda}(\mathbf{x}, \mathbf{t}) \right)^{\frac{\alpha+\beta}{\alpha+\beta-1}} (\tau(\mathbf{t}))^{\frac{1}{1-\alpha-\beta}} \Delta \mathbf{t}, \quad \mathbf{x} \in \Omega,$$

then

$$\int_{\Omega} [G'(|f(\mathbf{x})|)]^{\alpha} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha} \omega(\mathbf{x}) \Delta \mathbf{x} \leq N_{\Omega} \left[G \left(\int_{\Omega} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}}. \quad (3.16)$$

Similarly, if $f \in \mathcal{L}_{\rho^{\lambda-1}(b)}^{\alpha+\beta}(\Omega^{\kappa^{\lambda-1}}, \tau, \lambda)$ and

$$N_{\Omega^{\kappa^{\lambda-1}}}^* := \left[\int_{\Omega^{\kappa^{\lambda-1}}} [G'(\vartheta^*(\mathbf{x}))]^{\frac{\alpha(\alpha+\beta-1)}{\beta}} \omega^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau^{-\frac{\alpha}{\beta}}(\mathbf{x}) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} < \infty,$$

where

$$\vartheta^*(\mathbf{x}) := \int_{\bar{\Omega}_{\mathbf{x}}} |H_{\lambda}(\mathbf{x}, \mathbf{t})|^{\frac{\alpha+\beta}{\alpha+\beta-1}} (\tau(\mathbf{t}))^{\frac{1}{1-\alpha-\beta}} \Delta \mathbf{t}, \quad \mathbf{x} \in \Omega,$$

then,

$$\begin{aligned} & \int_{\Omega^{\kappa^{\lambda-1}}} [G'(|f(\mathbf{x})|)]^{\alpha} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha} \omega(\mathbf{x}) \Delta \mathbf{x} \\ & \leq N_{\Omega^{\kappa^{\lambda-1}}}^* \left[G \left(\int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}}. \end{aligned} \quad (3.17)$$

If $f \in \mathcal{L}_a^{\alpha+\beta}(\Omega_c, \tau, \lambda) \cap \mathcal{L}_{\rho^{\lambda-1}(b)}^{\alpha+\beta}(\bar{\Omega}_c, \tau, \lambda)$ and $N_{\Omega_c}, N_{\bar{\Omega}_c}^*$ are finite for all $c \in \Omega^{\kappa^{\lambda-1}}$ is such that $\Omega^{\kappa^{\lambda-1}} = \Omega_c \cup \bar{\Omega}_c$, then

$$\begin{aligned} & \int_{\Omega^{\kappa^{\lambda-1}}} [G'(|f(\mathbf{x})|)]^{\alpha} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha} \omega(\mathbf{x}) \Delta \mathbf{x} \\ & \leq N_{\Omega_c} \left[G \left(\int_{\Omega_c} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}} \\ & \quad + N_{\bar{\Omega}_c}^* \left[G \left(\int_{\bar{\Omega}_c} \left| \frac{\partial^{\lambda} f(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right|^{\alpha+\beta} \tau(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}}. \end{aligned} \quad (3.18)$$

Remark 3.8 Inequality (3.16) is the same as inequality given in [21, Theorem 2.9], if we take $n = 1$ and $G(x) = |x|^{\gamma}$ for $\gamma > 1$. If $\lambda_1 = 1$, then inequalities (3.16) and (3.17) reduce to inequalities given in [22, Theorems 3.1 and 3.2], respectively.

We examine Corollary 3.2 further in the case when $G(x) = |x|^{\frac{\alpha+\beta}{\alpha}}$. We obtain the following corollary.

Corollary 3.3 *Let $\omega, \tau \in \mathcal{W}(\Omega)$, and $f : \Omega \rightarrow (-R, R)$ be such that*

$$\int_{\Omega} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha+\beta} \tau(x) \Delta x < R.$$

If $f \in \mathcal{L}_a^{\alpha+\beta}(\Omega, \tau, \lambda)$ and

$$L_{\Omega} := \left[\int_{\Omega} \left(\int_{\Omega_x} H_{\lambda}^{\frac{\alpha+\beta}{\alpha+\beta-1}}(x, t) \tau^{\frac{1}{1-\alpha-\beta}}(t) \Delta t \right)^{\alpha+\beta-1} \omega^{\frac{\alpha+\beta}{\beta}}(x) \tau^{-\frac{\alpha}{\beta}}(x) \Delta x \right]^{\frac{\beta}{\alpha+\beta}} < \infty,$$

then

$$\int_{\Omega} |f(x)|^{\beta} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha} \omega(x) \Delta x \leq \left(\frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{\alpha+\beta}} L_{\Omega} \int_{\Omega} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha+\beta} \tau(x) \Delta x. \quad (3.19)$$

Similarly, if $f \in \mathcal{L}_{\rho^{\lambda-1}(b)}^{\alpha+\beta}(\Omega^{\kappa^{\lambda-1}}, \tau, \lambda)$ and

$$L_{\Omega^{\kappa^{\lambda-1}}}^* := \left[\int_{\Omega^{\kappa^{\lambda-1}}} \left(\int_{\bar{\Omega}_x} |H_{\lambda}(x, t)|^{\frac{\alpha+\beta}{\alpha+\beta-1}} \tau^{\frac{1}{1-\alpha-\beta}}(t) \Delta t \right)^{\alpha+\beta-1} \omega^{\frac{\alpha+\beta}{\beta}}(x) \tau^{-\frac{\alpha}{\beta}}(x) \Delta x \right]^{\frac{\beta}{\alpha+\beta}}$$

is finite, then

$$\begin{aligned} & \int_{\Omega^{\kappa^{\lambda-1}}} |f(x)|^{\beta} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha} \omega(x) \Delta x \\ & \leq \left(\frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{\alpha+\beta}} L_{\Omega^{\kappa^{\lambda-1}}}^* \int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha+\beta} \tau(x) \Delta x. \end{aligned} \quad (3.20)$$

If $f \in \mathcal{L}_a^{\alpha+\beta}(\Omega_c, \tau, \lambda) \cap \mathcal{L}_{\rho^{\lambda-1}(b)}^{\alpha+\beta}(\bar{\Omega}_c, \tau, \lambda)$, L_{Ω_c} and $L_{\bar{\Omega}_c}^$ are finite for $c \in \Omega^{\kappa^{\lambda-1}}$ is such that $\Omega^{\kappa^{\lambda-1}} = \Omega_c \cup \bar{\Omega}_c$, then*

$$\begin{aligned} & \int_{\Omega^{\kappa^{\lambda-1}}} |f(x)|^{\beta} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha} \omega(x) \Delta x \\ & \leq \left(\frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{\alpha+\beta}} \left[L_{\Omega_c} \int_{\Omega_c} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha+\beta} \tau(x) \Delta x \right. \\ & \quad \left. + L_{\bar{\Omega}_c}^* \int_{\bar{\Omega}_c} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha+\beta} \tau(x) \Delta x \right]. \end{aligned} \quad (3.21)$$

Furthermore, if we choose $c \in \Omega^{\kappa^{\lambda-1}}$ is such that $|L_{\Omega_c} - L_{\bar{\Omega}_c}^| \rightarrow \min$, then (3.21) reduces to*

$$\begin{aligned} & \int_{\Omega^{\kappa^{\lambda-1}}} |f(x)|^{\beta} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha} \omega(x) \Delta x \\ & \leq \left(\frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{\alpha+\beta}} \max\{L_{\Omega_c}, L_{\bar{\Omega}_c}^*\} \int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^{\lambda} f(x)}{\Delta x^{\lambda}} \right|^{\alpha+\beta} \tau(x) \Delta x. \end{aligned} \quad (3.22)$$

Remark 3.9 The results in Corollary 3.3 when $n = 1, \lambda = 1, \Omega = [a, b]_{\mathbb{T}}, f(a) = 0$ or/and $f(b) = 0$ yield many known results. Namely, if we set $a = 0, \alpha = \beta = 1, \omega = \tau \equiv 1$, then inequality (3.21) becomes

$$\begin{aligned} & \int_0^b |f(x)| |f^\Delta(x)| \Delta x \\ & \leq \frac{\sqrt{2}}{2} \left[\left(\int_0^c x \Delta x \right)^{1/2} \int_0^c |f^\Delta(x)|^2 \Delta x + \left(\int_c^b (b-x) \Delta x \right)^{1/2} \int_c^b |f^\Delta(x)|^2 \Delta x \right] \end{aligned} \quad (3.23)$$

for all $c \in [0, b]_{\mathbb{T}}$. Since $x \leq \frac{1}{2}(x^2)^\Delta$, then inequality (3.23) reduces to

$$\int_0^b |f(x)| |f^\Delta(x)| \Delta x \leq \frac{c}{2} \int_0^c |f^\Delta(x)|^2 \Delta x + \frac{b-c}{2} \int_c^b |f^\Delta(x)|^2 \Delta x \quad (3.24)$$

for all $c \in [0, b]_{\mathbb{T}}$. If $\mathbb{T} = \mathbb{R}$, then we can choose $c = b/2$ and inequality (3.24) yields inequality (1.1).

Next, inequalities (3.19) and (3.20) are the same as the ones given in [20, Theorems 2.4 and 2.5], respectively, if we take $n = 1$.

Remark 3.10 When $\mathbb{T} = \mathbb{R}$, from Corollary 3.3 we obtain many known results:

1. If $n = 2$ and $\lambda = (1, 1)$, then inequalities (3.19) and (3.20) imply the results of Pachpatte given in [17, Theorems 1 and 4], respectively.
2. Let $n = 2, \lambda = (\lambda_1, \lambda_2), \alpha = \beta = 1$, and $\omega = \tau \equiv 1$; then inequality (3.19) reduces to [26, Theorem 2.1].

The following result which can be proved in view of Corollary 3.2 and the well-known inequality of arithmetic and geometric means:

$$\prod_{j=1}^k a_j \leq \left(\frac{1}{k} \sum_{j=1}^k a_j \right)^k, \quad a_j \geq 0 \quad \text{for } j \in [1, k]_{\mathbb{N}}. \quad (3.25)$$

Corollary 3.4 Let $k \in \mathbb{N}$, $\alpha_j, \beta_j > 0$ be such that $\alpha_j + \beta_j > 1/k$, $G_j \in \mathcal{G}_R^{1,1}$, and ω_j, τ_j be in $\mathcal{W}(\Omega)$, and $f_j : \Omega \rightarrow (-R, R)$ be such that $\int_\Omega \left| \frac{\partial^\lambda f_j(x)}{\Delta x^\lambda} \right|^{k(\alpha+\beta)} \tau_j(x) \Delta x < R$ for $j \in [1, k]_{\mathbb{N}}$.

If $f_j \in \mathcal{L}_a^{k(\alpha_j+\beta_j)}(\Omega, \tau_j, \lambda)$ for all $j \in [1, k]_{\mathbb{N}}$, and

$$T_{\Omega_j} := \left[\int_\Omega [G'_j(\eta_j(x))]^{\frac{\alpha_j(k\alpha_j+k\beta_j-1)}{\beta_j}} \omega_j^{\frac{k(\alpha_k+\beta_i)}{\beta_j}}(x) \tau_j^{-\frac{\alpha_j}{\beta_j}}(x) \Delta x \right]^{\frac{\beta_j}{\alpha_j+\beta_j}} < \infty,$$

where

$$\eta_j(x) := \int_{\Omega_x} \left(H_\lambda(x, t) \right)^{\frac{k(\alpha_j+\beta_j)}{k(\alpha_j+\beta_j)-1}} (\tau_j(t))^{\frac{1}{1-k(\alpha_j+\beta_i)}} \Delta t, \quad x \in \Omega,$$

then

$$\begin{aligned} & \int_\Omega \prod_{j=1}^k [G'_j(|f_j(x)|)]^{\alpha_j} \left| \frac{\partial^\lambda f_j(x)}{\Delta x^\lambda} \right|^{\alpha_j} \omega_j(x) \Delta x \\ & \leq \frac{1}{k} \sum_{j=1}^k T_{\Omega_j} \left[G_j \left(\int_\Omega \left| \frac{\partial^\lambda f_j(x)}{\Delta x^\lambda} \right|^{k(\alpha_j+\beta_j)} \tau_j(x) \Delta x \right) \right]^{\frac{\alpha_j}{\alpha_j+\beta_j}}. \end{aligned} \quad (3.26)$$

If $f_j \in \mathcal{L}_{\rho^{\lambda-1}(b)}^{k(\alpha+\beta)}(\Omega^{\kappa^{\lambda-1}}, \tau_j, \lambda)$ for all $j \in [1, k]_{\mathbb{N}}$, and

$$T_{\Omega_j}^* := \left[\int_{\Omega} [G'_j(\eta_j^*(x))]^{\frac{\alpha_j(k\alpha_j+k\beta_j-1)}{\beta_j}} \omega_j^{\frac{k(\alpha_k+\beta_i)}{\beta_j}}(x) \tau_j^{-\frac{\alpha_j}{\beta_j}}(x) \Delta x \right]^{\frac{\beta_j}{\alpha_j+\beta_j}} < \infty,$$

where

$$\eta_j^*(x) := \int_{\Omega_x} |H_{\lambda}(x, t)|^{\frac{k(\alpha_j+\beta_j)}{k(\alpha_j+\beta_j)-1}} (\tau_j(t))^{\frac{1}{1-k(\alpha_j+\beta_i)}} \Delta t, \quad x \in \Omega^{\kappa^{\lambda-1}},$$

then

$$\begin{aligned} & \int_{\Omega^{\kappa^{\lambda-1}}} \prod_{j=1}^k [G'_j(|f_j(x)|)]^{\alpha_j} \left| \frac{\partial^{\lambda} f_j(x)}{\Delta x^{\lambda}} \right|^{\alpha_j} \omega_j(x) \Delta x \\ & \leq \frac{1}{k} \sum_{j=1}^k T_{\Omega_j}^* \left[G_j \left(\int_{\Omega^{\kappa^{\lambda-1}}} \left| \frac{\partial^{\lambda} f_j(x)}{\Delta x^{\lambda}} \right|^{k(\alpha_j+\beta_j)} \tau_j(x) \Delta x \right) \right]^{\frac{\alpha_j}{\alpha_j+\beta_j}}. \end{aligned} \quad (3.27)$$

Proof By using inequality (3.25), we obtain

$$\begin{aligned} & \prod_{j=1}^k [G'_j(|f_j(x)|)]^{\alpha_j} \left| \frac{\partial^{\lambda} f_j(x)}{\Delta x^{\lambda}} \right|^{\alpha_j} \omega_j(x) \\ & = \left(\prod_{j=1}^k [G'_j(|f_j(x)|)]^{\alpha_j k} \left| \frac{\partial^{\lambda} f_j(x)}{\Delta x^{\lambda}} \right|^{\alpha_j k} \omega_j^k(x) \right)^{\frac{1}{k}} \\ & \leq \frac{1}{k} \sum_{j=1}^k [G'_j(|f_j(x)|)]^{\alpha_j k} \left| \frac{\partial^{\lambda} f_j(x)}{\Delta x^{\lambda}} \right|^{\alpha_j k} \omega_j^k(x). \end{aligned}$$

Applying inequalities (3.16) and (3.17) we arrive at (3.26) and (3.27). \square

The following theorem generalizes Rozanova's inequality [3, Theorem 2.5] to functions of several variables on time scales.

Theorem 3.6 Let $F \in \mathcal{H}_{\infty}^m$, ϕ_i be convex, non-negative, and increasing on $[0, \infty)$, and $\varphi_i : \Omega \rightarrow \mathbb{R}$ be such that $\frac{\partial^1 \varphi_i(x)}{\Delta x^1}$ is non-negative with $\frac{\partial^{k_j} \varphi_i(x)}{\Delta_j x_j^{k_j}}|_{x_j=a_j} = 0$, where $k_j \in [0, \lambda_j - 1]_{\mathbb{N}}$, $j \in [1, n]_{\mathbb{N}}$, and $i \in [1, m]_{\mathbb{N}}$. For any $f_i \in C_{rd}^{n\lambda}(\Omega)$ is such that $\frac{\partial^{k_j} f_i(x)}{\Delta_j x_j^{k_j}}|_{x_j=a_j} = 0$ for all $k_j \in [0, \lambda_j - 1]_{\mathbb{N}}$, $j \in [1, n]_{\mathbb{N}}$, and $i \in [1, m]_{\mathbb{N}}$, we have

$$\begin{aligned} & \int_{\Omega} \left[\sum_{i=1}^m D_i F \left(\varphi_1(x) \phi_1 \left(\frac{|f_1(x)|}{\varphi_1(x)} \right), \dots, \varphi_m(x) \phi_m \left(\frac{|f_m(x)|}{\varphi_m(x)} \right) \right) \right. \\ & \quad \times \frac{\partial^1 \varphi_i(x)}{\Delta x^1} \phi_i \left(H_{\lambda}(b, a) \frac{|\frac{\partial^{\lambda} f_i(x)}{\Delta x^{\lambda}}|}{\frac{\partial^1 \varphi_i(x)}{\Delta x^1}} \right) \Big] \Delta x \\ & \leq F \left(\int_{\Omega} \frac{\partial^1 \varphi_1(x)}{\Delta x^1} \phi_1 \left(H_{\lambda}(b, a) \frac{|\frac{\partial^{\lambda} f_1(x)}{\Delta x^{\lambda}}|}{\frac{\partial^1 \varphi_1(x)}{\Delta x^1}} \right) \Delta x, \dots, \right. \\ & \quad \left. \int_{\Omega} \frac{\partial^1 \varphi_m(x)}{\Delta x^1} \phi_m \left(H_{\lambda}(b, a) \frac{|\frac{\partial^{\lambda} f_m(x)}{\Delta x^{\lambda}}|}{\frac{\partial^1 \varphi_m(x)}{\Delta x^1}} \right) \Delta x \right). \end{aligned} \quad (3.28)$$

Proof As in the proof of Theorem 3.2, for all $\mathbf{x} \in \Omega$, and all $i \in [1, m]_{\mathbb{N}}$, we have

$$|f_i(\mathbf{x})| \leq \int_{\Omega_{\mathbf{x}}} H_{\lambda}(\mathbf{x}, \mathbf{t}) \left| \frac{\partial^{\lambda} f_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}} \right| \Delta \mathbf{t} := y_i(\mathbf{x}).$$

We see that

$$y_i(\mathbf{x}) = \int_{\Omega_{\mathbf{x}}} H_{\lambda}(\mathbf{x}, \mathbf{t}) \frac{\partial^{\lambda} y_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}} \Delta \mathbf{t} \leq H_{\lambda}(\mathbf{b}, \mathbf{a}) \int_{\Omega_{\mathbf{x}}} \frac{\partial^{\lambda} y_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}} \Delta \mathbf{t}, \quad \mathbf{x} \in \Omega.$$

By using the fact that $\phi_i, i \in [1, m]_{\mathbb{N}}$, are non-negative, and increasing on $[0, \infty)$, we get

$$\begin{aligned} \phi_i \left(\frac{|f_i(\mathbf{x})|}{\varphi_i(\mathbf{x})} \right) &\leq \phi_i \left(\frac{y_i(\mathbf{x})}{\varphi_i(\mathbf{x})} \right) \\ &\leq \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \frac{\int_{\Omega_{\mathbf{x}}} \frac{\partial^{\lambda} y_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}} \Delta \mathbf{t}}{\int_{\Omega_{\mathbf{x}}} \frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1} \Delta \mathbf{t}} \right) \\ &= \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \int_{\Omega_{\mathbf{x}}} \frac{\frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1} \frac{\partial^{\lambda} y_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}}}{\frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1}} \Delta \mathbf{t} \right), \end{aligned}$$

which, in view of Jensen's inequality [24], we obtain

$$\phi_i \left(\frac{|f_i(\mathbf{x})|}{\varphi_i(\mathbf{x})} \right) \leq \frac{1}{\varphi_i(\mathbf{x})} \int_{\Omega_{\mathbf{x}}} \frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1} \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \frac{\frac{\partial^{\lambda} y_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}}}{\frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1}} \right) \Delta \mathbf{t} \quad (3.29)$$

for $\mathbf{x} \in \Omega$ and $i \in [1, m]_{\mathbb{N}}$.

Setting

$$W_i(\mathbf{t}) := \frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1} \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \frac{\frac{\partial^{\lambda} y_i(\mathbf{t})}{\Delta \mathbf{t}^{\lambda}}}{\frac{\partial^1 \varphi_i(\mathbf{t})}{\Delta \mathbf{t}^1}} \right), \quad \mathbf{t} \in \Omega_{\mathbf{x}}, \quad i \in [1, m]_{\mathbb{N}}.$$

Thus, (3.29) implies that

$$\varphi_i(\mathbf{x}) \phi_i \left(\frac{|f_i(\mathbf{x})|}{\varphi_i(\mathbf{x})} \right) \leq \int_{\Omega_{\mathbf{x}}} W_i(\mathbf{t}) \Delta \mathbf{t}, \quad \mathbf{x} \in \Omega, \quad i \in [1, m]_{\mathbb{N}}. \quad (3.30)$$

Since $D_i F$ for $i \in [1, m]_{\mathbb{N}}$, are non-negative, increasing in each variable on $(0, \infty)$ and (3.30), we get

$$\begin{aligned} &\int_{\Omega} \left[\sum_{i=1}^m D_i F \left(\varphi_1(\mathbf{x}) \phi_1 \left(\frac{|f_1(\mathbf{x})|}{\varphi_1(\mathbf{x})} \right), \dots, \varphi_m(\mathbf{x}) \phi_m \left(\frac{|f_m(\mathbf{x})|}{\varphi_m(\mathbf{x})} \right) \right) \right. \\ &\quad \times \left. \frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \frac{\frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}}}{\frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1}} \right) \right] \Delta \mathbf{x} \\ &\leq \int_{\Omega} \left[\sum_{i=1}^m D_i F \left(\int_{\Omega_{\mathbf{x}}} W_1(\mathbf{t}) \Delta \mathbf{t}, \dots, \int_{\Omega_{\mathbf{x}}} W_m(\mathbf{t}) \Delta \mathbf{t} \right) W_i(\mathbf{x}) \right] \Delta \mathbf{x}. \end{aligned}$$

Using (3.1), we obtain (3.28). \square

Remark 3.11 [3, Theorem 2.5] is a special case of Theorem 3.6 when $\mathbb{T} = \mathbb{R}$.

The following theorem is an interesting generalization of Theorem 3.3.

Theorem 3.7 *Let $G \in \mathcal{G}_R^{1,m}$ and $\omega_i, \tau_i \in \mathcal{W}(\Omega)$ for $i \in [1, m]_{\mathbb{N}}$ be such that*

$$P := \left[\int_{\Omega} \left(\sum_{i=1}^m D_i G(\nu_1(\mathbf{x}), \dots, \nu_m(\mathbf{x})) \omega_i^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau_i^{-\frac{\alpha}{\beta}}(\mathbf{x}) \right) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} < \infty,$$

where

$$\nu_i(\mathbf{x}) = \left(\int_{\Omega_{\mathbf{x}}} (\tau_i(t))^{\frac{1}{1-\alpha-\beta}} \Delta t \right)^{\frac{\alpha(\alpha+\beta-1)}{\beta}}, \quad \mathbf{x} \in \Omega.$$

Further, for $i \in [1, m]_{\mathbb{N}}$, let ϕ_i be convex, non-negative, and increasing on $[0, \infty)$, and $\varphi_i : \Omega \rightarrow \mathbb{R}$ be such that $\frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1}$ is non-negative with $\frac{\partial^{k_j} \varphi_i(\mathbf{x})}{\Delta_j x_j^{k_j}}|_{x_j=a_j} = 0$, where

$k_j \in [0, \lambda_j - 1]_{\mathbb{N}}$, $j \in [1, n]_{\mathbb{N}}$. If $f_i \in C_{rd}^{\lambda}(\Omega)$ is such that $\frac{\partial^{k_j} f_i(\mathbf{x})}{\Delta_j x_j^{k_j}}|_{x_j=a_j} = 0$ for all $k_j \in [0, \lambda_j - 1]_{\mathbb{N}}$, $j \in [1, n]_{\mathbb{N}}$, and $i \in [1, m]_{\mathbb{N}}$, and

$$\int_{\Omega} \left| \frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \right) \right|^{\alpha+\beta} \tau_i(\mathbf{x}) \Delta \mathbf{x} < R$$

for all $i \in [1, m]_{\mathbb{N}}$, then we have

$$\begin{aligned} & \int_{\Omega} \left[\sum_{i=1}^m D_i G \left(\varphi_1^{\alpha}(\mathbf{x}) \phi_1^{\alpha} \left(\frac{|f_1(\mathbf{x})|}{\varphi_1(\mathbf{x})} \right), \dots, \varphi_m^{\alpha}(\mathbf{x}) \phi_m^{\alpha} \left(\frac{|f_m(\mathbf{x})|}{\varphi_m(\mathbf{x})} \right) \right) \right. \\ & \quad \times \left. \left| \frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_i \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \left| \frac{\partial^{\lambda} f_i(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \right) \right|^{\alpha} \omega_i(\mathbf{x}) \right] \Delta \mathbf{x} \\ & \leq P \left[G \left(\int_{\Omega} \left| \frac{\partial^1 \varphi_1(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_1 \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \left| \frac{\partial^{\lambda} f_1(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \right) \right|^{\alpha+\beta} \tau_1(\mathbf{x}) \Delta \mathbf{x}, \dots, \right. \right. \\ & \quad \left. \left. \int_{\Omega} \left| \frac{\partial^1 \varphi_m(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_m \left(H_{\lambda}(\mathbf{b}, \mathbf{a}) \left| \frac{\partial^{\lambda} f_m(\mathbf{x})}{\Delta \mathbf{x}^{\lambda}} \right| \right) \right|^{\alpha+\beta} \tau_m(\mathbf{x}) \Delta \mathbf{x} \right) \right]^{\frac{\alpha}{\alpha+\beta}}. \end{aligned} \quad (3.31)$$

Proof As in the proof of Theorem 3.6, using (3.30) and Hölder's inequality with indices $(\alpha + \beta)/(\alpha + \beta - 1)$ and $(\alpha + \beta)$, we obtain

$$\begin{aligned} \varphi_i(\mathbf{x}) \phi_i \left(\frac{|f_i(\mathbf{x})|}{\varphi_i(\mathbf{x})} \right) & \leq \int_{\Omega_{\mathbf{x}}} W_i(t) \Delta t \\ & \leq \left(\int_{\Omega_{\mathbf{x}}} (\tau_i(t))^{\frac{1}{1-\alpha-\beta}} \Delta t \right)^{\frac{\alpha+\beta-1}{\alpha+\beta}} \left(\int_{\Omega_{\mathbf{x}}} (W_i(t))^{\alpha+\beta} \tau_i(t) \Delta t \right)^{\frac{1}{\alpha+\beta}} \end{aligned}$$

for $\mathbf{x} \in \Omega$ and $i \in [1, m]_{\mathbb{N}}$. Hence

$$\begin{aligned} & \varphi_i^{\alpha}(\mathbf{x}) \phi_i^{\alpha} \left(\frac{|f_i(\mathbf{x})|}{\varphi_i(\mathbf{x})} \right) \\ & \leq \left(\int_{\Omega_{\mathbf{x}}} (\tau_i(t))^{\frac{1}{1-\alpha-\beta}} \Delta t \right)^{\frac{\alpha(\alpha+\beta-1)}{\alpha+\beta}} \left(\int_{\Omega_{\mathbf{x}}} (W_i(t))^{\alpha+\beta} \tau_i(t) \Delta t \right)^{\frac{\alpha}{\alpha+\beta}} \\ & = (\nu_i(\mathbf{x}))^{\frac{\beta}{\alpha+\beta}} \left(\int_{\Omega_{\mathbf{x}}} (W_i(t))^{\alpha+\beta} \tau_i(t) \Delta t \right)^{\frac{\alpha}{\alpha+\beta}}, \quad \mathbf{x} \in \Omega, \quad i \in [1, m]_{\mathbb{N}}. \end{aligned} \quad (3.32)$$

Thus, by $G \in \mathcal{G}_R^{1,m}$ and (3.32), we get

$$\begin{aligned} & \int_{\Omega} \left[\sum_{i=1}^m D_i G \left(\varphi_1^\alpha(\mathbf{x}) \phi_1^\alpha \left(\frac{|f_1(\mathbf{x})|}{\varphi_1(\mathbf{x})} \right), \dots, \varphi_m^\alpha(\mathbf{x}) \phi_m^\alpha \left(\frac{|f_m(\mathbf{x})|}{\varphi_m(\mathbf{x})} \right) \right) \right. \\ & \quad \times \left. \left| \frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_i \left(H_\lambda(\mathbf{b}, \mathbf{a}) \frac{\left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|}{\frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1}} \right) \right|^\alpha \omega_i(\mathbf{x}) \right] \Delta \mathbf{x}, \\ & \leq \int_{\Omega} \left[\sum_{i=1}^m (D_i G(\nu_1(\mathbf{x}), \dots, \nu_m(\mathbf{x})))^{\frac{\beta}{\alpha+\beta}} \left(D_i G \left(\int_{\Omega_{\mathbf{x}}} (W_1(\mathbf{t}))^{\alpha+\beta} \tau_1(\mathbf{t}) \Delta \mathbf{t}, \dots, \right. \right. \right. \\ & \quad \left. \left. \left. \int_{\Omega_{\mathbf{x}}} (W_m(\mathbf{t}))^{\alpha+\beta} \tau_m(\mathbf{t}) \Delta \mathbf{t} \right) \right)^{\frac{\alpha}{\alpha+\beta}} \left| \frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1} \phi_i \left(H_\lambda(\mathbf{b}, \mathbf{a}) \frac{\left| \frac{\partial^\lambda f_i(\mathbf{x})}{\Delta \mathbf{x}^\lambda} \right|}{\frac{\partial^1 \varphi_i(\mathbf{x})}{\Delta \mathbf{x}^1}} \right) \right|^\alpha \omega_i(\mathbf{x}) \right] \Delta \mathbf{x}, \end{aligned}$$

which, applying Hölder's inequality with indices $(\alpha + \beta)/\beta$ and $(\alpha + \beta)/\alpha$, yields

$$\begin{aligned} & \leq \left[\int_{\Omega} \left(\sum_{i=1}^m D_i G(\nu_1(\mathbf{x}), \dots, \nu_m(\mathbf{x})) \omega_i^{\frac{\alpha+\beta}{\beta}}(\mathbf{x}) \tau_i^{-\frac{\alpha}{\beta}}(\mathbf{x}) \right) \Delta \mathbf{x} \right]^{\frac{\beta}{\alpha+\beta}} \\ & \quad \times \left[\int_{\Omega} \left(\sum_{i=1}^m D_i G \left(\int_{\Omega_{\mathbf{x}}} (W_1(\mathbf{t}))^{\alpha+\beta} \tau_1(\mathbf{t}) \Delta \mathbf{t}, \dots, \right. \right. \right. \\ & \quad \left. \left. \left. \int_{\Omega_{\mathbf{x}}} (W_m(\mathbf{t}))^{\alpha+\beta} \tau_m(\mathbf{t}) \Delta \mathbf{t} \right) (W_i(\mathbf{x}))^{\alpha+\beta} \tau_i(\mathbf{x}) \right) \Delta \mathbf{x} \right]^{\frac{\alpha}{\alpha+\beta}}, \end{aligned}$$

which completes the proof by applying (3.1). \square

4 Applications

In this section, we use Opial-type inequalities to establish Lyapunov-type inequalities for the following half-linear dynamic equation

$$\frac{\partial^1}{\Delta \mathbf{x}^1} \left(r(\mathbf{x}) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right) + s(\mathbf{x}) y^\sigma(\mathbf{x}) = 0 \quad \text{on} \quad D = [a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2}, \quad (4.1)$$

where $\mathbb{T}_1, \mathbb{T}_2$ be arbitrary time scales, $r(\mathbf{x}), s(\mathbf{x})$ be weights on D , and $y^\sigma(\mathbf{x}) := y(\sigma_1(x_1), \sigma_2(x_2))$ for all $\mathbf{x} = (x_1, x_2) \in D$.

We define

$$D_{\mathbf{x}} = \{t \in D : \mathbf{a} \leq t \leq \mathbf{x}\}, \quad \mathbf{a} = (a_1, a_2),$$

$$\bar{D}_{\mathbf{x}} = \{t \in D : \mathbf{x} \leq t \leq \mathbf{b}\}, \quad \mathbf{b} = (b_1, b_2),$$

$$S(\mathbf{x}) = \int_{\bar{D}_{\mathbf{x}}} s(t) \Delta t,$$

and

$$S^*(x_1) = (b_2 - a_2) [\sup_{\mathbf{x} \in D} S(\mathbf{x})] (b_1 - x_1) \quad \text{for} \quad \mathbf{x} = (x_1, x_2) \in D.$$

For $\mathbf{c} = (c_1, c_2) \in D$, we set

$$K_1(\mathbf{c}) = \left[\int_{D_{\mathbf{c}}} \left(\int_{D_{\mathbf{x}}} r^{-1}(\mathbf{x}) \Delta t \right) (S^*(x_1) + S(\mathbf{x}))^2 r^{-1}(\mathbf{x}) \Delta \mathbf{x} \right]^{\frac{1}{2}},$$

$$L_1(\mathbf{c}) = \left[\int_{\bar{D}_{\mathbf{c}}} \left(\int_{\bar{D}_{\mathbf{x}}} r^{-1}(\mathbf{x}) \Delta \mathbf{t} \right) (S^*(x_1) + S(\mathbf{x}))^2 r^{-1}(\mathbf{x}) \Delta \mathbf{x} \right]^{\frac{1}{2}},$$

and

$$M = \frac{\sqrt{2}}{2} \max\{K_1(\mathbf{c}), L_1(\mathbf{c})\},$$

where $\mathbf{c} \in D$ is such that $D = D_{\mathbf{c}} \cup \bar{D}_{\mathbf{c}}$ and $|K_1(\mathbf{c}) - L_1(\mathbf{c})| \rightarrow \min$.

Now, fix $x_1 \in [a_1, b_1]_{\mathbb{T}_1}$, and put

$$N_1(x_1, c'_2) = \left[\int_{a_2}^{c'_2} \left(\int_{a_2}^{x_2} r^{-1}(\mathbf{x}) \Delta t_2 \right) (S^*(x_1) + S(\mathbf{x}))^2 r^{-1}(\mathbf{x}) \Delta x_2 \right]^{\frac{1}{2}},$$

$$N_2(x_1, c'_2) = \left[\int_{c'_2}^{b_2} \left(\int_{x_2}^{b_2} r^{-1}(\mathbf{x}) \Delta t_2 \right) (S^*(x_1) + S(\mathbf{x}))^2 r^{-1}(\mathbf{x}) \Delta x_2 \right]^{\frac{1}{2}},$$

and

$$N(x_1) = \frac{\sqrt{2}}{2} \max\{N_1(x_1, c'_2), N_2(x_1, c'_2)\},$$

where $c'_2 \in [a_2, b_2]_{\mathbb{T}}$ is such that $|N_1(x_1, c'_2) - N_2(x_1, c'_2)| \rightarrow \min$.

Theorem 4.1 Suppose that y is a nontrivial solution of (4.1) such that

$$y(\mathbf{x})|_{x_j=a_j} = y(\mathbf{x})|_{x_j=b_j} = \frac{\partial y(\mathbf{x})}{\Delta_i x_i} \Big|_{x_j=a_j} = \frac{\partial y(\mathbf{x})}{\Delta_i x_i} \Big|_{x_j=b_j} = 0, \quad i, j = 1, 2, \quad (4.2)$$

then

$$2M + \sup_{x_1 \in [a_1, b_1]_{\mathbb{T}_1}} [\mu_1(x_1) N(x_1)] \geq 1. \quad (4.3)$$

Proof Multiplying (4.1) by $y^\sigma(\mathbf{x})$ and integrating both sides of (4.1) with respect to \mathbf{x} over D we get

$$\int_D \frac{\partial^1}{\Delta \mathbf{x}^1} \left(r(\mathbf{x}) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right) y^\sigma(\mathbf{x}) \Delta \mathbf{x} = - \int_D s(\mathbf{x}) (y^\sigma(\mathbf{x}))^2 \Delta \mathbf{x}. \quad (4.4)$$

By integrating by parts each side of (4.4), we have

$$\begin{aligned} & \int_D \frac{\partial^1}{\Delta \mathbf{x}^1} \left(r(\mathbf{x}) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right) y^\sigma(\mathbf{x}) \Delta \mathbf{x} \\ &= - \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial}{\Delta_1 x_1} \left(r(\mathbf{x}) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right) \frac{\partial y(\sigma_1(x_1), x_2)}{\Delta_2 x_2} \Delta x_2 \Delta x_1 \\ &= \int_D r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta \mathbf{x}, \end{aligned}$$

where, we have used boundary conditions (4.2), and

$$\int_D s(\mathbf{x}) (y^\sigma(\mathbf{x}))^2 \Delta \mathbf{x} = - \int_D \frac{\partial^1 S(\mathbf{x})}{\Delta \mathbf{x}^1} (y^\sigma(\mathbf{x}))^2 \Delta \mathbf{x} = - \int_D S(\mathbf{x}) \frac{\partial^1 y^2(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta \mathbf{x},$$

where, we have used

$$y^2(\sigma_1(x_1), a_2) = y^2(\sigma_1(x_1), b_2) = y(\sigma_1(x_1), b_2) = \frac{\partial y^2(a_1, x_2)}{\Delta_2 x_2} = \frac{\partial y^2(b_1, x_2)}{\Delta_2 x_2} = 0.$$

Therefore, we obtain

$$\int_D r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta \mathbf{x} = \left| \int_D S(\mathbf{x}) \frac{\partial^1 y^2(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta \mathbf{x} \right|. \quad (4.5)$$

Since

$$\frac{\partial^1 y^2(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta \mathbf{x} = \left(y^\sigma(\mathbf{x}) + y(x_1, \sigma_2(x_2)) \right) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} + \left(\mu_1(x_1) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} + 2y(\mathbf{x}) \right) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1}$$

it follows that

$$\begin{aligned} \left| \int_D S(\mathbf{x}) \frac{\partial^1 y^2(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta \mathbf{x} \right| &\leq \left| \int_D S(\mathbf{x}) y^\sigma(\mathbf{x}) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Delta \mathbf{x} \right| \\ &\quad + \left| \int_D S(\mathbf{x}) y(x_1, \sigma_2(x_2)) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Delta \mathbf{x} \right| \\ &\quad + \int_D S(\mathbf{x}) \mu_1(x_1) \left| \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \right| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \\ &\quad + 2 \int_D S(\mathbf{x}) |y(\mathbf{x})| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x}. \end{aligned} \quad (4.6)$$

We have

$$\begin{aligned} &\left| \int_D S(\mathbf{x}) y(x_1, \sigma_2(x_2)) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Delta \mathbf{x} \right| \\ &\leq \int_{a_1}^{b_1} S^*(x_1) \left| \int_{a_2}^{b_2} y(x_1, \sigma_2(x_2)) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Delta x_2 \right| \Delta x_1 \\ &= \int_{a_1}^{b_1} S^*(x_1) \left| \int_{a_2}^{b_2} y(x_1, x_2) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta x_2 \right| \Delta x_1 \\ &= \int_D S^*(x_1) |y(\mathbf{x})| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x}, \end{aligned} \quad (4.7)$$

where, we have used

$$\begin{aligned} \int_{a_2}^{b_2} y(x_1, \sigma_2(x_2)) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Delta x_2 &= y(x_1, x_2) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Big|_{x_2=a_2}^{x_2=b_2} - \int_{a_2}^{b_2} y(x_1, x_2) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta x_2 \\ &= - \int_{a_2}^{b_2} y(x_1, x_2) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta x_2. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \left| \int_D S(\mathbf{x}) y^\sigma(\mathbf{x}) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \Delta \mathbf{x} \right| \\
& \leq \int_D S^*(x_1) |y(\sigma_1(x_1), x_2)| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \\
& \leq \int_D S^*(x_1) |y(\mathbf{x} + \mu_1(x_1)) \frac{\partial y(\mathbf{x})}{\Delta_1 x_1}| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \\
& \leq \int_D S^*(x_1) |y(\mathbf{x})| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \\
& + \int_D S^*(x_1) \mu_1(x_1) \left| \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \right| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x}.
\end{aligned} \tag{4.8}$$

From (4.6), (4.7), and (4.8), we get

$$\begin{aligned}
& \left| \int_D S(\mathbf{x}) \frac{\partial^1 y^2(\mathbf{x})}{\Delta \mathbf{x}^1} \Delta \mathbf{x} \right| \\
& \leq 2 \int_D (S^*(x_1) + S(\mathbf{x})) |y(\mathbf{x})| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \\
& + \int_D (S^*(x_1) + S(\mathbf{x})) \mu_1(x_1) \left| \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \right| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x}.
\end{aligned}$$

Applying inequality (3.22) with $n = 2, \alpha = \beta = 1, \lambda = \mathbf{1}, \omega(\mathbf{x}) = S^*(x_1) + S(\mathbf{x})$, and $\tau(\mathbf{x}) = r(\mathbf{x})$, we get

$$\int_D (S^*(x_1) + S(\mathbf{x})) |y(\mathbf{x})| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta \mathbf{x} \leq M \int_D r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta \mathbf{x}. \tag{4.9}$$

Applying inequality (3.22) with $n = 1, \alpha = \beta = 1, \lambda = 1, \omega(\mathbf{x}) = S^*(x_1) + S(\mathbf{x})$, and $\tau(\mathbf{x}) = r(\mathbf{x})$, we have

$$\int_{a_2}^{b_2} (S^*(x_1) + S(\mathbf{x})) \left| \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \right| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta x_2 \leq N(x_1) \int_{a_2}^{b_2} r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta x_2.$$

This implies that

$$\begin{aligned}
& \int_{a_1}^{b_1} \int_{a_2}^{b_2} (S^*(x_1) + S(\mathbf{x})) \mu_1(x_1) \left| \frac{\partial y(\mathbf{x})}{\Delta_1 x_1} \right| \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right| \Delta x_2 \Delta x_1 \\
& \leq \int_{a_1}^{b_1} \mu_1(x_1) N(x_1) \int_{a_2}^{b_2} r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta x_2 \Delta x_1 \\
& \leq \left(\sup_{x_1 \in [a_1, b_1]_{\tau_1}} [\mu_1(x_1) N(x_1)] \right) \int_D r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta \mathbf{x}.
\end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.5), we have

$$\int_D r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta \mathbf{x} \leq \left(2M + \sup_{x_1 \in [a_1, b_1]_{\tau_1}} [\mu_1(x_1) N(x_1)] \right) \int_D r(\mathbf{x}) \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right)^2 \Delta \mathbf{x}.$$

Then, we get

$$2M + \sup_{x_1 \in [a_1, b_1]_{\mathbb{T}_1}} [\mu_1(x_1)N(x_1)] \geq 1.$$

□

In Theorem 4.1 if $r(\mathbf{x}) \equiv 1$, then we have the following result.

Corollary 4.1 *Suppose that y is a nontrivial solution of*

$$\frac{\partial^1}{\Delta \mathbf{x}^1} \left(\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right) + s(\mathbf{x})y^\sigma(\mathbf{x}) = 0, \quad \mathbf{x} \in D$$

which satisfies boundary conditions (4.2), then

$$2 \max\{K_2(\mathbf{c}), L_2(\mathbf{c})\} + \sup_{x_1 \in [a_1, b_1]_{\mathbb{T}_1}} \left(\mu_1(x_1) \max\{N_3(x_1, c'_2), N_4(x_1, c'_2)\} \right) \geq \sqrt{2},$$

where

$$K_2(\mathbf{c}) = \left[\int_{D_c} (x_1 - a_1)(x_2 - a_2)[S^*(x_1) + S(\mathbf{x})]^2 \Delta \mathbf{x} \right]^{\frac{1}{2}},$$

$$L_2(\mathbf{c}) = \left[\int_{\bar{D}_c} (b_1 - x_1)(b_2 - x_2)[S^*(x_1) + S(\mathbf{x})]^2 \Delta \mathbf{x} \right]^{\frac{1}{2}},$$

with $\mathbf{c} \in D$ is such that $D = D_c \cup \bar{D}_c$ and $|K_2(\mathbf{c}) - L_2(\mathbf{c})| \rightarrow \min$, and

$$N_3(x_1, c'_2) = \left[\int_{a_2}^{c'_2} (x_2 - a_2)[S^*(x_1) + S(\mathbf{x})]^2 \Delta x_2 \right]^{\frac{1}{2}},$$

$$N_4(x_1, c'_2) = \left[\int_{c'_2}^{b_2} (b_2 - x_2)[S^*(x_1) + S(\mathbf{x})]^2 \Delta x_2 \right]^{\frac{1}{2}},$$

for $x_1 \in [a_1, b_1]_{\mathbb{T}_1}$, $c'_2 \in [a_2, b_2]_{\mathbb{T}}$ is such that $|N_3(x_1, c'_2) - N_4(x_1, c'_2)| \rightarrow \min$.

As a special case of Corollary 4.1, when $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, we see that if $\mathbf{c} = (\frac{a_1+b_1}{2}, b_2)$ and $\mathbf{c} = (a_1, \frac{a_2+b_2}{2})$ are in $[a_1, b_1] \times [a_2, b_2]$, then $|K_2(\mathbf{c}) - L_2(\mathbf{c})| = 0$, and $\mu_1(x_1) = 0$, we have the following result.

Corollary 4.2 *Suppose that y is a nontrivial solution of*

$$\frac{\partial^1}{\partial \mathbf{x}^1} \left(\frac{\partial^1 y(\mathbf{x})}{\partial \mathbf{x}^1} \right) + s(\mathbf{x})y(\mathbf{x}) = 0, \quad \mathbf{x} \in [a_1, b_1] \times [a_2, b_2]$$

which satisfies (4.2), then

$$\int_D (x_1 - a_1)(x_2 - a_2)[S^*(x_1) + S(\mathbf{x})]^2 d\mathbf{x} \geq 1.$$

Remark 4.1 We see that the sufficient condition that equation (4.1) does not have a nontrivial solution y such that

$$y(\mathbf{x})|_{x_j=a_j} = y(\mathbf{x})|_{x_j=b_j} = \frac{\partial y(\mathbf{x})}{\Delta_i x_i} \Big|_{x_j=a_j} = \frac{\partial y(\mathbf{x})}{\Delta_i x_i} \Big|_{x_j=b_j} = 0, \quad i, j = 1, 2,$$

can be obtained from Theorem 4.1.

Now, we consider the following half-linear delay dynamic equation

$$\frac{\partial^1}{\Delta \mathbf{x}^1} \left(r(\mathbf{x}) \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right) + s(\mathbf{x}) y(\theta(\mathbf{x})) = 0 \quad \text{on } D, \quad (4.11)$$

where $\theta(\mathbf{x}) = (\theta_1(x_1), \theta_2(x_2))$ for all $\mathbf{x} = (x_1, x_2) \in D$, $\theta_j : \mathbb{T}_j \rightarrow \mathbb{T}_j$, $\theta_j(x_j) \leq x_j$, and $\lim_{x_j \rightarrow \infty} \theta_j(x_j) = \infty$ for $j = 1, 2$.

Corollary 4.3 *Suppose that y is a nontrivial solution of equation (4.11) which satisfies boundary conditions (4.2) and $\frac{\partial y(\mathbf{x})}{\Delta_1 x_1}, \frac{\partial y(\mathbf{x})}{\Delta_2 x_2}$, do not change sign in $(a_1, b_1)_{\mathbb{T}_1} \times (a_2, b_2)_{\mathbb{T}_2}$, then*

$$2M + \sup_{x_1 \in [a_1, b_1]_{\mathbb{T}_1}} [\mu_1(x_1) N(x_1)] \geq 1.$$

Proof Since $\frac{\partial y(\mathbf{x})}{\Delta_1 x_1}, \frac{\partial y(\mathbf{x})}{\Delta_2 x_2}$ do not change sign in $(a_1, b_1)_{\mathbb{T}_1} \times (a_2, b_2)_{\mathbb{T}_2}$ we can assume that (4.11) has a solution y satisfying $\frac{\partial y(\mathbf{x})}{\Delta_1 x_1} > 0, \frac{\partial y(\mathbf{x})}{\Delta_2 x_2} > 0$. Therefore, since $\theta(\mathbf{x}) \leq \mathbf{x}$, that $y(\theta(\mathbf{x})) \leq y^\sigma(\mathbf{x})$. The remainder of the proof is similar to the proof of Theorem 4.1 and hence omitted. \square

Finally, we give an upper bound of the solutions of the following integro-partial dynamic equation

$$\frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} = \zeta \left(\mathbf{x}, \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1}, I(y(\mathbf{x})) \right), \quad \mathbf{x} \in \Omega, \quad (4.12)$$

with the initial conditions $y(\mathbf{x})|_{x_j=a_j} = 0$ for all $j \in [1, n]_{\mathbb{N}}$, where

$$I(y(\mathbf{x})) = \int_{\Omega_{\mathbf{x}}} \Phi \left(\mathbf{t}, y(\mathbf{t}), \frac{\partial^1 y(\mathbf{t})}{\Delta \mathbf{t}^1} \right) \Delta \mathbf{t}.$$

Theorem 4.2 *Suppose that $\Phi(\mathbf{x}, y(\mathbf{x}), \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1}) \leq \omega(\mathbf{x}) |y(\mathbf{x})|^\beta \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right|^\alpha$, where $\omega \in W(\Omega)$, and*

$$\left| \zeta \left(\mathbf{x}, \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1}, I(y(\mathbf{x})) \right) \right| \leq w_1(\mathbf{x}) + w_2(\mathbf{x}) \left| \frac{\partial^1 y(\mathbf{x})}{\Delta \mathbf{x}^1} \right|^\gamma + w_3(\mathbf{x}) |I(y(\mathbf{x}))|, \quad (4.13)$$

where $\gamma \in (0, 1)$, w_1, w_2 , and w_3 are non-negative on Ω , $w_2(\mathbf{x}) < 1$ for all $\mathbf{x} \in \Omega$. If equation (4.12) has a solution $y \in \mathcal{L}_a^{\alpha+\beta}(\Omega, 1, \mathbf{1})$, then

$$y(\mathbf{x}) \leq \int_{\Omega_{\mathbf{x}}} [A^{1-\alpha-\beta}(\mathbf{t}) + (1-\alpha-\beta)B(\mathbf{t}) \text{Vol}(\Omega_{\mathbf{t}})]^{\frac{1}{1-\alpha-\beta}} \Delta \mathbf{t} \quad (4.14)$$

as long as the right-hand side integral exists, where

$$B(\mathbf{x}) = \sup_{\mathbf{t} \in \Omega_{\mathbf{x}}} \frac{w_3(\mathbf{t})K(\mathbf{t})}{1-w_2(\mathbf{t})}, \quad A(\mathbf{x}) = \sup_{\mathbf{t} \in \Omega_{\mathbf{x}}} \frac{w_1(\mathbf{t}) + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}}{1-w_2(\mathbf{t})},$$

$\text{Vol}(\Omega_{\mathbf{t}})$ is the volume of a rectangular region $\Omega_{\mathbf{t}}$, and

$$K(\mathbf{x}) = \left(\frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{\alpha+\beta}} \left(\int_{\Omega_{\mathbf{x}}} (\text{Vol}(\Omega_{\mathbf{t}}))^{\alpha+\beta-1} \omega^{\frac{\alpha+\beta}{\beta}}(\mathbf{t}) \Delta \mathbf{t} \right)^{\frac{\beta}{\alpha+\beta}} \quad \text{for } \mathbf{x} \in \Omega.$$

Proof By applying inequality (3.19) with $\lambda = 1, \tau \equiv 1$, we have

$$\int_{\Omega_x} \omega(t) |y(t)|^\beta \left| \frac{\partial^1 y(t)}{\Delta t^1} \right|^\alpha \Delta t \leq K(x) \int_{\Omega_x} \left| \frac{\partial^1 y(t)}{\Delta t^1} \right|^{\alpha+\beta} \Delta t \quad (4.15)$$

for $x \in \Omega$. We consider the function $f(x) = x^\gamma - x - (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}$ for $\gamma \in (0, 1)$ and $x \geq 0$. We see that $f(x)$ obtains its maximum at $x = \gamma^{\frac{1}{1-\gamma}}$ and $f_{\max} = 0$. Then, we have

$$\left| \frac{\partial^1 y(x)}{\Delta x^1} \right|^\gamma \leq \left| \frac{\partial^1 y(x)}{\Delta x^1} \right| + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}, \quad x \in \Omega. \quad (4.16)$$

Substituting (4.15) and (4.16) into (4.13), we obtain

$$(1 - w_2(x)) \left| \frac{\partial^1 y(x)}{\Delta x^1} \right| \leq w_1(x) + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} + w_3(x)K(x) \int_{\Omega_x} \left| \frac{\partial^1 y(t)}{\Delta t^1} \right|^{\alpha+\beta} \Delta t$$

for $x \in \Omega$. This implies that

$$\left| \frac{\partial^1 y(x)}{\Delta x^1} \right| \leq \frac{w_1(x) + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}}{1 - w_2(x)} + \frac{w_3(x)K(x)}{1 - w_2(x)} \int_{\Omega_x} \left| \frac{\partial^1 y(t)}{\Delta t^1} \right|^{\alpha+\beta} \Delta t \quad (4.17)$$

for $x \in \Omega$. Let $s \in \Omega$ be arbitrary, but fixed; then inequality (4.17) gives

$$\left| \frac{\partial^1 y(t)}{\Delta t^1} \right| \leq A(s) + B(s) \int_{\Omega_t} \left| \frac{\partial^1 y(u)}{\Delta u} \right|^{\alpha+\beta} \Delta u, \quad t \in \Omega_s. \quad (4.18)$$

Next, let $R(t)$ be the right-hand side of (4.18); then

$$\frac{\partial^1 R(t)}{\Delta t^1} = B(s) \left| \frac{\partial^1 y(t)}{\Delta t^1} \right|^{\alpha+\beta} \leq B(s) R^{\alpha+\beta}(t), \quad t \in \Omega_s,$$

where $R|_{t_j=a_j} = A(s)$ for all $j \in [1, n]_{\mathbb{N}}$. Since

$$\int_{\Omega_z} \frac{1}{R^{\alpha+\beta}(t)} \frac{\partial^1 R(t)}{\Delta t^1} \Delta t \geq \frac{1}{1-\alpha-\beta} [R^{1-\alpha-\beta}(z) - A^{1-\alpha-\beta}(s)], \quad z \in \Omega_s,$$

then

$$\frac{1}{1-\alpha-\beta} [R^{1-\alpha-\beta}(z) - A^{1-\alpha-\beta}(s)] \leq B(s) \text{Vol}(\Omega_z), \quad z \in \Omega_s.$$

Therefore,

$$\left| \frac{\partial^1 y(z)}{\Delta z^1} \right| \leq R(z) \leq [A^{1-\alpha-\beta}(s) + (1-\alpha-\beta)B(s) \text{Vol}(\Omega_z)]^{\frac{1}{1-\alpha-\beta}}, \quad z \in \Omega_s.$$

In the above inequality replacing z by s and integrating both sides with respect to s over Ω_x for $x \in \Omega$ we obtain (4.14). \square

Acknowledgements The author would like to express his deepest gratitude to Assoc. Prof. Dinh Thanh Duc, Prof. Vu Kim Tuan and Nguyen Du Vi Nhan for their comments and suggestions to improve this paper.

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